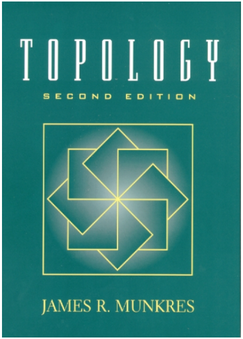
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 14. The Order Topology—Proofs of Theorems



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Proof. We confirm that the definition of basis of a topology is satisfied. If a_0 is the least element of X then $a_0 \in [a_0, b)$ for all $b \neq a_0$. If b_0 is the greatest element of X then $b_0 \in (a, b_0]$ for all $a \neq b_0$.

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Theorem 14.B

Theorem 14.B. Let X be a set with a simple order relation. The open rays form a subbasis for the order topology $\mathcal{T}$ on X.

Proof. Let $\mathcal{S}$ be the set of all open rays. As observed above, the open rays are in fact open sets in the order topology, so $\mathcal{S} \subset \mathcal{T}$ and the topology generated by $\mathcal{S}$ is a subset of $\mathcal{T}$ as well (Lemma 31.1).

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