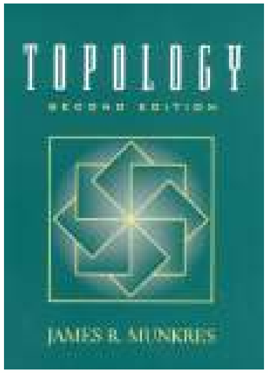
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 15. The Product Topology on $X \times Y$—Proofs of Theorems



Theorem 15.1

Theorem 15.1. If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection $\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

Proof. Let $W \subset X \times Y$ be an open set and let $(x, y) \in W$. By the definition of product topology, there is a basis element $U \times V$, where U is open in X and V is open in Y, such that $(x, y) \in U \times V \subset W$. So $x \in U$ and $y \in V$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, then there are open $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subset U$ and $y \in C \subset V$. Notice that $B \times C$ is an element of the basis for the product topology and so is open and $B \times C \in \mathcal{D}$. That is, $(x, y) \in B \times C \subset W$ where $B \times C \in \mathcal{D}$, so by Theorem 13.2, $\mathcal{D}$ is a basis for the product topology. □

Theorem 15.2

Theorem 15.2. The set

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$. Let $\mathcal{T}'$ be the topology generated by set $\mathcal{S}$. For open sets $U \subset X$ and $V \subset Y$, we have that $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ are elements of the basis for the product topology $\mathcal{T}$ (and so are open in $\mathcal{T}$) and hence $\mathcal{S} \subset \mathcal{T}$. So arbitrary unions (and finite intersections) of elements of $\mathcal{S}$ are in $\mathcal{T}$. Therefore, by Lemma 13.1, $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times v$ for $\mathcal{T}$ is of the form $U \times V = (U \times Y) \cap (X \times V) = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ (a finite intersection of elements of $\mathcal{S}$) and so is in the topology $\mathcal{T}'$ generated by $\mathcal{S}$. That is, $\mathcal{T} \subset \mathcal{T}'$ and hence $\mathcal{T} = \mathcal{T}'$. So the collection of all unions of finite intersections of $\mathcal{S}$ is $\mathcal{T}$ and $\mathcal{S}$ is a subbasis for the product topology $\mathcal{T}$. □