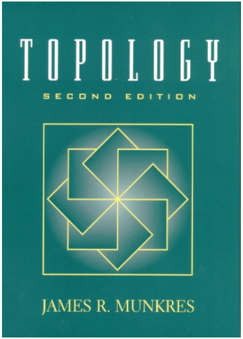
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 15. The Product Topology on $X \times Y$—Proofs of Theorems



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$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$. Let $\mathcal{T}'$ be the topology generated by set $\mathcal{S}$.

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