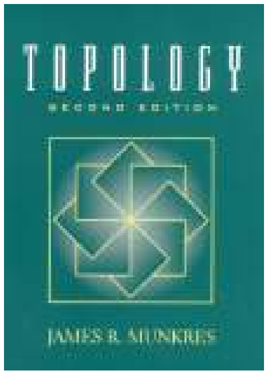
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 16. The Subspace Topology—Proofs of Theorems



Lemma 16.1

Lemma 16.1. If $\mathcal{B}$ is a basis for the topology of X then the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

Proof. Let U be open in X so that $U \cap Y \in \mathcal{B}_Y$. Let $y \in U \cap Y$. Then since $\mathcal{B}$ is a basis for the topology of X, there is (open) $B \in \mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap y \subset U \cap Y$. Then by Lemma 13.2, $\mathcal{B}_Y$ is a basis for the subspace topology on Y. □

Lemma 16.2

Lemma 16.2. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof. Let U be open in Y. Then $U = Y \cap V$ for some set V open in X. Since Y and V are both open in X, then $Y \cap V = U$ is open in X. □

Lemma 16.3

Lemma 16.3. If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof. Let $U \times V$ be a basis element for the product topology on $X \times Y$. Then $(U \times V) \cap (A \times B)$ is a basis element for the subspace topology on $A \times B$. Now $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$. Since $U \cap A$ and $V \cap B$ are open relative to A and B, respectively, then $(U \cap A) \times (V \cap B)$ is a basis element for the product topology on $A \times B$. So the basis for the subspace topology on $A \times B$ is a subset of the basis for the product topology on $A \times B$. Conversely, a basis element for the product topology on $A \times B$ is of the form $(U \cap A) \times (V \cap B)$ where U and v are open in X and Y, respectively, and from the equality above, this is a basis element for the subspace topology on $A \times B$. So the basis for the product topology on $A \times B$ is a subset of the basis for the subspace topology on $A \times B$. So the bases are the same and, as claimed, the topologies are the same. □

Theorem 16.4

Theorem 16.4. Let X be an ordered set in the order topology. Let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the subspace topology on Y.

Proof. By Theorem 14.B, the set of all open rays form a subbasis for the order topology on X. Then the set $\mathcal{B}_S = \{(a,+\infty) \cap Y, Y \cap (-\infty, a) \mid a \in X\}$ is a subbasis for the subspace topology on Y. Since Y is convex then for $a \in Y$ we have $(a,+\infty) \cap Y = \{a \in Y \mid x > a\}$ and $(-\infty, a) \cap Y = \{x \in Y \mid x < a\}$ and each of these is an open ray in Y. If $a \notin Y$ then these two sets are either all of Y or are $\varnothing$. In all cases, each is open in the order topology and so the subspace topology is a subset of the subspace topology.

Conversely, any open ray of Y equals the intersection of an open ray of X with Y and so is open in the subspace topology on Y. Since the open rays of Y are a subbasis for the order topology on Y by Theorem 14.B, this topology is a subset of the subspace topology. Therefore, the subspace topology on Y is the same as the order topology on Y. □