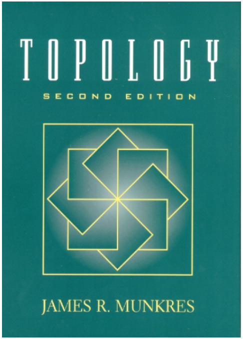
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 16. The Subspace Topology—Proofs of Theorems



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Lemma 16.1. If $\mathcal{B}$ is a basis for the topology of X then the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

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Lemma 16.2. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof. Let U be open in Y. Then $U = Y \cap V$ for some set V open in X. Since Y and V are both open in X, then $Y \cap V = U$ is open in X. □

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Proof. Let $U \times V$ be a basis element for the product topology on $X \times Y$. Then $(U \times V) \cap (A \times B)$ is a basis element for the subspace topology on $A \times B$.

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Theorem 16.4

Theorem 16.4. Let X be an ordered set in the order topology. Let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the subspace topology on Y.

Proof. By Theorem 14.B, the set of all open rays form a subbasis for the order topology on X. Then the set $\mathcal{B}_S = \{(a, +\infty) \cap Y, Y \cap (-\infty, a) \mid a \in X\}$ is a subbasis for the subspace topology on Y.

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