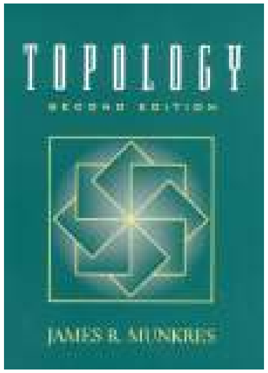
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions
Section 17. Closed Sets and Limit Points—Proofs of Theorems



Theorem 17.1

Theorem 17.1. Let X be a topological space. Then the following conditions hold:

(1) $\varnothing$ and X are closed.
(2) Arbitrary intersections of closed sets are closed.
(3) Finite unions of closed sets are closed

Proof. Since the compliments of $\varnothing$ and X are X and $\varnothing$, respectively, then by definition of closed, both $\varnothing$ and X are closed (since X and $\varnothing$ are open) and (1) follows.

Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we have be DeMorgan's law (see Munkres' page 11), $X \setminus \cap_{\alpha \in J} A_\alpha = \cup_{\alpha \in J}(X \setminus A_\alpha)$. Since the sets $X \setminus A_\alpha$ are open by definition, the right side of this equation is a union of open sets and so is open. Therefore the left hand side is open an so, by definition, its compliment $\cap_{\alpha \in J} A_\alpha$ is closed, as claimed in (2).

Theorem 17.1 (continued)

Theorem 17.1. Let X be a topological space. Then the following conditions hold:

(1) $\varnothing$ and X are closed.
(2) Arbitrary intersections of closed sets are closed.
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Proof (continued). If A_i is closed for $i = 1, 2, \ldots, n$, then again by DeMorgan's Law, $X \setminus \cup_{i=1}^n A_i = \cap_{i=1}^n (X \setminus A_i)$. The set on the right side is a finite intersection of open sets and is therefore open. So the left hand side is open and its compliment, $\cup_{i=1}^n A_i$, is closed, as claimed in (3). □

Theorem 17.2

Theorem 17.2. Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof. Suppose $A = C \cap Y$ where C is closed in Y. Then $X \setminus C$ is open in X and so $(x \setminus X) \cap Y$ is open in Y (by the definition of the subspace topology). But $(X \setminus C) \cap Y = Y \setminus A$ (the compliment of A is Y), so $Y \setminus A$ is open in Y and hence A is closed in Y.

Conversely, suppose that A is closed in Y. Then $Y \setminus A$ is open in Y (by definition of "A is closed in Y"). So, by definition of "open in $Y \subset X$," there is open U in X such that $Y \setminus A = Y \cap U$. Next, $X \setminus U$ is closed in X and $A = Y \cap (X \setminus U)$ so that A is the intersection of Y and a closed set $X \setminus U$ of X. □

Lemma 17.A

Lemma 17.A. Let A be a subset of topological space X. Then A is open if and only if $A = \text{Int}(A)$. A is closed if and only if $A = \overline{A}$.

Proof. If $A = \text{Int}(A)$ then, since $\text{Int}(A)$ is open, A is open. If A is open then, by the definition of $\text{Int}(A)$ as the union of all open subsets contained in A, we have $A \subset \text{Int}(A)$. As commented above, $\text{Int}(A) \subset A$ so if A is open then $A = \text{Int}(A)$.

If $A = \overline{\text{A}}$ then, since A is close, $\overline{A}$ is closed. If A is closed then, by the definition of $\overline{A}$ as the intersection of all closed sets containing A, we have $\overline{A} \subset A$. As commented above, $A \subset \overline{A}$ so if A is closed the $A = \overline{A}$. □

Theorem 17.4

Theorem 17.4. Let Y be a subspace of X. Let $A \subset Y$ and denote the closure of A in X as $\overline{A}$. Then the closure of A in Y equals $\overline{A} \cap Y$.

Proof. Let B denote the closure of A in Y. Since $\overline{A}$ is closed in X, then $\overline{A} \cap Y$ is closed in Y by Theorem 17.2. Since $\overline{A} \cap Y$ contains A (we are given $A \subset Y$) and since, by definition, B equals the intersection of all closed subsets of Y containing A, so we must have $B \subset \overline{A} \cap Y$.

On the other hand,m B is closed in Y. Hence by Theorem 17.2, $B = C \cap Y$ for some closed C in X. Then C is a closed set of X containing A (because $A \subset B \subset C$). Now $\overline{A}$ is the intersection of all closed sets in X containing A, so $\overline{A} \subset C$. Then $\overline{A} \cap Y \subset C \cap Y = B$. Therefore, $\overline{A} \cap Y = B$, as claimed. □

Theorem 17.5

Theorem 17.5 Let A be a subset of the topological space X.

(a) Then $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

(b) Supposing the topology of X is given a basis , then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof. (a) Consider the contrapositive: "$x \notin \overline{A}$ if and only if there is a neighborhood of x that does not intersect A." If $x \notin \overline{A}$ then the set $U = X \setminus \overline{A}$ is a neighborhood of x which does not intersect A, as claimed. Conversely, if there is a neighborhood U of x which does not intersect A, then $X \setminus U$ is a closed set containing A. By definition of the closure $\overline{A}$, the set $X \setminus U$ must contain $\overline{A}$. Since $x \in U$, then $x \notin \overline{A}$
(b) Suppose $x \in \overline{A}$. Then by part (a), every neighborhood of x intersects A. Then every basis element B containing x intersects A (since each B is open). Conversely, if every basis element containing x intersects A, then so does every neighborhood U of x because U contains a basis element that contains x. □

Theorem 17.6

Theorem 17.6 Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.

Proof. If $x \in A'$ then every neighborhood of x intersects A in a point different from x. Therefore, by Theorem 17.5(a), x belongs to $\overline{A}$. Hence $A' \subset \overline{A}$. Since $A \subset \overline{A}$, we have $A \cup A' \subset \overline{A}$.

Let $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$. If $x \notin A$ then, since $x \in \overline{A}$, every neighborhood U of x intersects A. Because $x \notin A$ then U must intersect A in a point different from x. Then $x \in A'$ so that $x \in A \cup A'$. Therefore, $\overline{A} \subset A \cup A'$ and hence $\overline{A} = A \cup A'$, as claimed. □

Theorem 17.7

Corollary 17.7. A subset of a topological space is closed if and only if it contains all its limit points.

Proof. The set A is closed if and only if $A = \overline{A}$ by Lemma 17.A. By Theorem 17.6, $\overline{A} = A \cup A'$, so $A = \overline{A}$ if and only if $A' \subset A$. □

Theorem 17.8

Theorem 17.8. Every finite point set in a Hausdorff space X is closed. In particular, singletons form closed sets in a Hausdorff space.

Proof. Consider the set $\{x_0\}$. If $x \in X$ where $x \neq x_0$ then, since X is a Hausdorff space, there are disjoint neighborhoods U of x and V of x_0. Since U does not intersect $\{x_0\}$, by Theorem 17.5(a), x is not in the closure of set $\{x_0\}$. Since $x \neq x_0$ is an arbitrary element of X, the only points of closure of $\{x_0\}$ is x_0 itself and so by Corollary 17.7 $\{x_0\}$ is a closed set. Now if we consider a finite point set, say $\{x_0, x_1, \dots, x_n\}$, then we simply write the set as $\{x_0\} \cup \{x_1\} \cup \cdots \cup \{x_n\}$, observe that each $\{x_i\}$ is closed, and apply Theorem 17.1 part (3). □

Theorem 17.9

Theorem 17.9. Let X be a space satisfying the "T_1 Axiom" (namely, that all finite point sets are closed). Let A be a subset of X. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof. Suppose every neighborhood of x intersects A in infinitely many points. Then every neighborhood of x intersects set A at a point other than x and so (by definition) x is a limit point of A.
Conversely, suppose that x is a limit point of A. ASSUME some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A \setminus \{x\}$ in finitely many points, say $\{x_1, x_2, \dots, x_m\}$ $= U \cap (A \setminus \{x\})$. The set $X \setminus \{x_1, x_2, \dots, x_m\}$ is open in X since by T_1 Axiom $\{x_1, x_2, \dots, x_m\}$ is closed. Then $U \cap (X \setminus \{x_1, x_2, \dots, x_m\})$ is a neighborhood of x that does not intersect the set $A \setminus \{x\}$. But this CONTRADICTS the hypothesis that x is a limit point of x. So the assumption that U intersects A in finitely many points is false. That is, any neighborhood of x must intersect A in infinitely many points. □

Theorem 17.10

Theorem 17.10. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Proof. Let x_n be a sequence of points of X that converges to x. If $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U is a neighborhood of x, then there is $N_1 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N_1$. So there is no $N_2 \in \mathbb{N}$ such that for $n \geq N_2$ we have $x_n \in V$ (since for $n \geq N_1$, $x_n \in U$ and $U \cap V = \varnothing$). That is, x_n does not converge to $y \neq x$ and x_n converges to at most one point in X. □