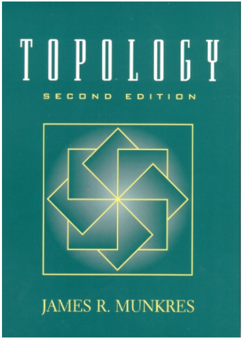
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 17. Closed Sets and Limit Points—Proofs of Theorems



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(1) $\varnothing$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed

Proof. Since the compliments of $\varnothing$ and X are X and $\varnothing$, respectively, then by definition of closed, both $\varnothing$ and X are closed (since X and $\varnothing$ are open) and (1) follows.

Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we have be DeMorgan's law (see Munkres' page 11), $X \setminus \cap_{\alpha \in J} A_\alpha = \cup_{\alpha \in J}(X \setminus A_\alpha)$.

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Proof (continued). If A_i is closed for $i = 1, 2, \ldots, n$, then again by DeMorgan's Law, $X \setminus \cup_{i=1}^{n} A_i = \cap_{i=1}^{n} (X \setminus A_i)$. The set on the right side is a finite intersection of open sets and is therefore open.

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Theorem 17.2. Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof. Suppose $A = C \cap Y$ where C is closed in Y. Then $X \setminus C$ is open in X and so $(x \setminus X) \cap Y$ is open in Y (by the definition of the subspace topology).

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Conversely, suppose that A is closed in Y. Then $Y \setminus A$ is open in Y (by definition of "A is closed in Y"). So, by definition of "open in $Y \subset X$," there is open U in X such that $Y \setminus A = Y \cap U$.

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Lemma 17.A

Lemma 17.A. Let A be a subset of topological space X. Then A is open if and only if $A = \text{Int}(A)$. A is closed if and only if $A = \overline{A}$.

Proof. If $A = \text{Int}(A)$ then, since $\text{Int}(A)$ is open, A is open. If A is open then, by the definition of $\text{Int}(A)$ as the union of all open subsets contained in A, we have $A \subset \text{Int}(A)$.

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If $A = \overline{A}$ then, since A is close, $\overline{A}$ is closed. If A is closed then, by the definition of $\overline{A}$ as the intersection of all closed sets containing A, we have $\overline{A} \subset A$.

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Theorem 17.4

Theorem 17.4. Let Y be a subspace of X. Let $A \subset Y$ and denote the closure of A in X as $\overline{A}$. Then the closure of A in Y equals $\overline{A} \cap Y$.

Proof. Let B denote the closure of A in Y. Since $\overline{A}$ is closed in X, then $\overline{A} \cap Y$ is closed in Y by Theorem 17.2.

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Proof. Let B denote the closure of A in Y. Since $\overline{A}$ is closed in X, then $\overline{A} \cap Y$ is closed in Y by Theorem 17.2. Since $\overline{A} \cap Y$ contains A (we are given $A \subset Y$) and since, by definition, B equals the intersection of all closed subsets of Y containing A, so we must have $B \subset \overline{A} \cap Y$.

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On the other hand,m B is closed in Y. Hence by Theorem 17.2, $B = C \cap Y$ for some closed C in X. Then C is a closed set of X containing A (because $A \subset B \subset C$).

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Theorem 17.5

Theorem 17.5 Let A be a subset of the topological space X.

(a) Then $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

(b) Supposing the topology of X is given a basis , then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

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Theorem 17.6

Theorem 17.6 Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.

Proof. If $x \in A'$ then every neighborhood of x intersects A in a point different from x. Therefore, by Theorem 17.5(a), x belongs to $\overline{A}$.

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Let $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$. If $x \notin A$ then, since $x \in \overline{A}$, every neighborhood U of x intersects A. Because $x \notin A$ then U must intersect A in a point different from x. Then $x \in A'$ so that $x \in A \cup A'$. Therefore, $\overline{A} \subset A \cup A'$ and hence $\overline{A} = A \cup A'$, as claimed. □

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Let $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$. If $x \notin A$ then, since $x \in \overline{A}$, every neighborhood U of x intersects A. Because $x \notin A$ then U must intersect A in a point different from x. Then $x \in A'$ so that $x \in A \cup A'$. Therefore, $\overline{A} \subset A \cup A'$ and hence $\overline{A} = A \cup A'$, as claimed. □

Theorem 17.7

Corollary 17.7. A subset of a topological space is closed if and only if it contains all its limit points.

Proof. The set A is closed if and only if $A = \overline{A}$ by Lemma 17.A. By Theorem 17.6, $\overline{A} = A \cup A'$, so $A = \overline{A}$ if and only if $A' \subset A$. □

Theorem 17.7

Corollary 17.7. A subset of a topological space is closed if and only if it contains all its limit points.

Proof. The set A is closed if and only if $A = \overline{A}$ by Lemma 17.A. By Theorem 17.6, $\overline{A} = A \cup A'$, so $A = \overline{A}$ if and only if $A' \subset A$. □

Theorem 17.8

Theorem 17.8. Every finite point set in a Hausdorff space X is closed. In particular, singletons form closed sets in a Hausdorff space.

Proof. Consider the set $\{x_0\}$. If $x \in X$ where $x \neq x_0$ then, since X is a Hausdorff space, there are disjoint neighborhoods U of x and V of x_0.

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Proof. Consider the set $\{x_0\}$. If $x \in X$ where $x \neq x_0$ then, since X is a Hausdorff space, there are disjoint neighborhoods U of x and V of x_0. Since U does not intersect $\{x_0\}$, by Theorem 17.5(a), x is not in the closure of set $\{x_0\}$. Since $x \neq x_0$ is an arbitrary element of X, the only points of closure of $\{x_0\}$ is x_0 itself and so by Corollary 17.7 $\{x_0\}$ is a closed set.

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Theorem 17.9

Theorem 17.9. Let X be a space satisfying the "T_1 Axiom" (namely, that all finite point sets are closed). Let A be a subset of X. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof. Suppose every neighborhood of x intersects A in infinitely many points. Then every neighborhood of x intersects set A at a point other than x and so (by definition) x is a limit point of A.

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Conversely, suppose that x is a limit point of A. ASSUME some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A \setminus \{x\}$ in finitely many points, say $\{x_1, x_2, \ldots, x_m\} = U \cap (A \setminus \{x\})$.

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Conversely, suppose that x is a limit point of A. ASSUME some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A \setminus \{x\}$ in finitely many points, say $\{x_1, x_2, \ldots, x_m\}$ $= U \cap (A \setminus \{x\})$. The set $X \setminus \{x_1, x_2, \ldots, x_m\}$ is open in X since by T_1 Axiom $\{x_1, x_2, \ldots, x_m\}$ is closed. Then $U \cap (X \setminus \{x_1, x_2, \ldots, x_m\})$ is a neighborhood of x that does not intersect the set $A \setminus \{x\}$. But this CONTRADICTS the hypothesis that x is a limit point of x.

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Proof. Let x_n be a sequence of points of X that converges to x. If $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U is a neighborhood of x, then there is $N_1 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N_1$.

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