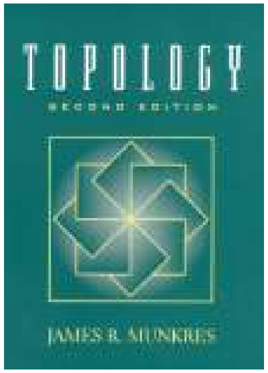
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 18. Continuous Functions—Proofs of Theorems



Lemma 18.A

Lemma 18.A. Let $f : X \to Y$, let $\mathcal{B}$ be a basis for the topology on Y, and let $\mathcal{S}$ be a subbasis for the topology on Y.

(1) f is continuous if $f^{-1}(B)$ is open in X for each $B \in \mathcal{B}$.

(2) f is continuous if $f^{-1}(S)$ is open in X for each $X \in \mathcal{S}$.

Proof. (1) Let $V \subset Y$ be open. Then (by definition of basis) there are $B_\alpha \in \mathcal{B}$ for $\alpha \in J$ such that $V = \cup_{\alpha \in J} B_\alpha$. Then $f^{-1}(V) = f^{-1}(\cup_{\alpha \in J} B_\alpha) = \cup_{\alpha \in J} f^{-1}(B_\alpha)$ is an open set in X by hypothesis. So each $f^{-1}(B_\alpha)$ is open in X and $f^{-1}(V)$ is open in X. Hence f is continuous.

Lemma 18.A (continued)

Lemma 18.A. Let $f : X \to Y$, let $\mathcal{B}$ be a basis for the topology on Y, and let $\mathcal{S}$ be a subbasis for the topology on Y.

(1) f is continuous if $f^{-1}(B)$ is open in X for each $B \in \mathcal{B}$.

(2) f is continuous if $f^{-1}(S)$ is open in X for each $X \in \mathcal{S}$.

Proof (continued). (2) Let $V \subset Y$ be open. Then (by the definition of subbasis) there are S_α^i for $\alpha \in J$, $i \in \mathbb{N}$ such that $V = \cup_{\alpha \in J}(S_\alpha^1 \cap S_\alpha^2 \cap \cdots \cap S_\alpha^{n_\alpha})$. Then

$$f^{-1}(V) = f^{-1}(\cup_{\alpha \in J}(S_\alpha^1 \cap S_\alpha^2 \cap \cdots \cap S_\alpha^{n_\alpha})) = \cup_{\alpha \in J} f^{-1}(S_\alpha^1 \cap S_\alpha^2 \cap \cdots \cap S_\alpha^{n_\alpha})$$
$$= \cup_{\alpha \in J}(f^{-1}(S_\alpha^1) \cap f^{-1}(S_\alpha^2) \cap \cdots \cap f^{-1}(S_\alpha^{n_\alpha}))$$

is open in X since each $f^{-1}(S_\alpha^i)$ is open in X by hypothesis and so $f^{-1}(S_\alpha^1) \cap f^{-1}(S_\alpha^2) \cap \cdots \cap f^{-1}(S_\alpha^{n_\alpha})$ is open for each $\alpha \in J$, and hence the union is open. So $f^{-1}(V)$ is open and f is continuous. □

Theorem 18.1

Theorem 18.1. Let X and Y be topological spaces. let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.

(2) For every subset Z of X, one has $f(\overline{A}) \subset \overline{f(A)}$.

(3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof. (1)⇒(2) Suppose f is continuous. Let $A \subset X$ and $x \in \overline{A}$. If $x \in A$ then $f(x) \in f(A) \subset \overline{f(A)}$. If $x \notin A$ then let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open in X and $x \in f^{-1}(V)$. By definition of $\overline{A}$, $f^{-1}(V)$ intersects A at some point $y \neq x$. So $f(y) \in V \cap f(A)$ (notice that $f(y) \neq f(x)$ since $f(x) \notin f(A)$). So $f(x) \in \overline{f(A)}$. So $f(x) \in \overline{f(A)}$ for any $x \in \overline{A}$ and hence $f(\overline{A}) \subset \overline{f(A)}$.

Theorem 18.1 (continued 1)

Theorem 18.1. Let X and Y be topological spaces. let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.

(2) For every subset Z of X, one has $f(\overline{A}) \subset \overline{f(A)}$.

(3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof (continued). (2)⇒(3) Let B be closed in Y and let $A = f^{-1}(B)$. Then $f(A) \subset B$ (f may not be onto B and so we may not have $f(A) = B$). So if $x \in \overline{A}$ then $f(x) \in f(\overline{A}) \subset \overline{f(A)}$ by hypothesis (2) and $\overline{f(A)} \subset \overline{B} = B$ since $f(A) \subset B$ and B is closed. Hence $f(x) \in B$ and $x \in f^{-1}(B) = A$. So $\overline{A} \subset A$ and (since $A \subset \overline{A}$) we have $A = \overline{A}$ so that $A = f^{-1}(B)$ is closed (by Lemma 17.A), as claimed.

Theorem 18.1 (continued 2)

Theorem 18.1. Let X and Y be topological spaces. let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.

(3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.

Proof (continued). (3)⇒(1) Let V be an open set in Y. Set $B = Y \setminus V$. Then

$$\begin{aligned} f^{-1}(B) &= f^{-1}(Y \setminus V) = f^{-1} \setminus f^{-1}(V) \text{ by Exercise 2.2(d)} \\ &= X \setminus f^{-1}(V) \text{ since } X \text{ is the domain of } f. \end{aligned}$$

Since V is open, B is closed in Y and so by hypothesis (3), $f^{-1}(B) = X \setminus f^{-1}(V)$ is closed in X and so $f^{-1}(V)$ is open. Therefore, by the definition of continuous function, f is continuous.

Theorem 18.1 (continued 3)

Theorem 18.1. Let X and Y be topological spaces. let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof (continued). (1)⇒(4) Let $x \in X$ and let V be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is open since f is continuous and $x \in U$. That is, $f(U) \subset V$, as claimed.

(4)⇒(1) Let V be an open set of Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and so by hypothesis (4) there is open U_x in X with $x \in U_x$ and $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. Then with such open U_x chosen for each $x \in f^{-1}(V)$ we have $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$ and hence $f^{-1}(V)$ is open. Therefore, by the definition of continuous function, f is continuous and (1) follows. □

Theorem 18.2(a,b,c)

Theorem 18.2. Rules for Constructing Continuous Functions. Let X, Y, and Z be topological spaces.

(a) (Constant Function) If $f : X \to Y$ maps all of X into a single point $y_0 \in Y$, then f is continuous.

(b) (Inclusion) if A is a subspace of X, the inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

Proof. (a) Let $f(x) = y_0$ for every $x \in X$. Let V be open in Y. Then $f^{-1}(V) = X$ if $y_0 \in V$ and $f^{-1}(V) = \varnothing$ if $y_0 \notin V$. In either case, $f^{-1}(V)$ is open and so f is continuous. (b) If U is open in X, then $j^{-1}(U) = U \cap A$ which is open in A (by definition of the subspace topology). (c) If U is open in Z then $g^{-1}(U)$ is open in Y since g is continuous and $f^{-1}(g^{-1}(U))$ is open in X since f is continuous. Now $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U) = f^{-1}(g^{-1}(U))$ and so $g \circ f$ is continuous. □

Theorem 18.2

Theorem 18.2. Rules for Constructing Continuous Functions.
Let X, Y, and Z be topological spaces.

(d) (Restricting the Domain) If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.

(e) (Restricting or Expanding the Range) let $f : X \to Y$ be continuous. If X is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the functions $h : X \to Z$ obtained by expanding the range of f is continuous.

(e) (Local Formulation of Continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.

Theorem 18.2(d, e, f) (continued 1)

Proof. (d) The function $f|_A$ equals the composition of the inclusion map $j : A \to Y$ (which is continuous by part (b)) and $f : X \to Y$ (which is continuous by hypothesis). So by part (c), $f|_A$ is continuous.
(e) Let $f : X \to Y$ be continuous and $f(X) \subset Z \subset Y$. Let B be open in Z. Then (by definition) $B = Z \cap U$ for some open U in Y. Then

$$\begin{aligned} g^{-1}(B) &= g^{-1}(Z \cap U) = g^{-1}(Z) \cap g^{-1}(U) \\ &= X \cap g^{-1}(U) \text{since } f(X) = g(X) \subset Z \\ &= g^{-1}(U) \\ &= f^{-1}(U) \text{ since } f(x) \in Y \text{ for some } x \in X \text{ implies } g(x) = f(x) \in \end{aligned}$$

Since f is continuous, $f^{-1}(U)$ is open in X and so $g^{-1}(U)$ is open in X. Therefore, g is continuous.
Now let $h : X \to Z \supset Y$ be as described. Then h is the composition of $f : X \times Y$ (which is continuous by hypothesis) and the inclusion map $j : Y \to Z$ (which is continuous by part (b)). So, by part (c), h is continuous.

Theorem 18.2(d, e, f) (continued 2)

Proof. (f) Suppose $X = \cup_{\alpha \in J} U_\alpha$ for open U_α in X where $f|_{U_\alpha}$ is continuous for each $\alpha \in J$. Let V be an open set in Y. Since $f^{-1}(V) \cap U_\alpha$ consists of $x \in X \cap U_\alpha = U_\alpha$ such that $f(x) \in V$ and $(f|_{U_\alpha})^{-1}(V)$ consists of $x \in U_\alpha$ such that $f(x) \in U_\alpha$, then $f^{-1}(V) \cap U_\alpha = (f|_{U_\alpha})^{-1}(V)$ for all $\alpha \in J$. Since $f|_{U_\alpha}$ is continuous by hypothesis, then this set is open in U_α and since U_α is open then (by Lemma 16.2) this set is open in X. Since $X = \cup_{\alpha \in J} U_\alpha$ then

$$f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap (\cup_{\alpha \in J} U_\alpha) = \cup_{\alpha \in J}(f^{-1}(V) \cap U_\alpha)$$

is open in X since each set in the union is open. Therefore (by definition) f is continuous. □

Theorem 18.3

Theorem 18.3. The Pasting Lemma for Closed Sets.
Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cup B$, then f and g combine (or "paste") to give a continuous function $h : X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Proof. Let C be closed in Y. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f is continuous by hypothesis then $f^{-1}(C)$ is closed in A, by Theorem 18.1 (the (1)⇒(3) part), and so $f^{-1}(C)$ is closed in X since A is closed (that is, $f^{-1}(C) = A \cap D$ for closed D in X, so $f^{-1}(C)$ is closed in X). Similarly, $g^{-1}(C)$ is closed in B and in X. Therefore $h^{-1}(C)$ is closed in X and so by Theorem 18.2 (the (3)⇒(1) part) h is continuous. □

Theorem 18.4

Theorem 18.4. Maps into Products.
Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$ where $f_1 : A \to X$ and $f_2 : Y \to B$. Then f is continuous if and only if the functions f_1 and f_2 are continuous.

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. Then for U open in X and V open in Y, we have $\pi_1^{-1}(U) = U \times T$ and $\pi_2^{-1}(V) = X \times V$ open in $X \times Y$ (by the definition of product topology; these are basis elements for the product topology on $X \times Y$). So π_1 and π_2 are continuous. Note that for each $a \in A$, $\pi_1(f(a)) = \pi_1((f_1(a), f_2(a)) = f_1(a)$ and $\pi_2(f(a)) = \pi_2((f_1(a), f_2(a)) = f_2(a)$. So $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$. Suppose f is continuous. Then, by Theorem 18.2 part (c), f_1 and f_2 are continuous.

Theorem 18.4 (continued)

Theorem 18.4. Maps into Products.
Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$ where $f_1 : A \to X$ and $f_2 : Y \to B$. Then f is continuous if and only if the functions f_1 and f_2 are continuous.

Proof (continued). Suppose f_1 and f_2 are continuous. Let $U \times V$ be a basis element for the product topology of $X \times Y$ (so U is open in X and V is open in Y). Now $a \in f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, or if and only if $f_1(a) \in U$ and $f_2(a) \in V$, or if and only if $a \in f_1^{-1}(U) \cap f_2^{-1}(V)$. That is, $f^{-1}(U \times V) = f_1(U) \cap f_2^{-1}(V)$. Since f_1 an df_2 are continuous then $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in X and so $f^{-1}(U \times V)$ is open in X. Since every open set in $X \times Y$ can be written as a union of basis elements by Lemma 13.1, say $\cup_{\alpha \in J} U_\alpha \times V_\alpha$, and $f^{-1}(\cup_{\alpha \in J} U_\alpha \times V_\alpha = \cup_{\alpha \in J} f^{-1}(U_\alpha \times V_\alpha)$, then the inverse image of any open set in $X \times Y$ is open in A. That is, f is continuous. □