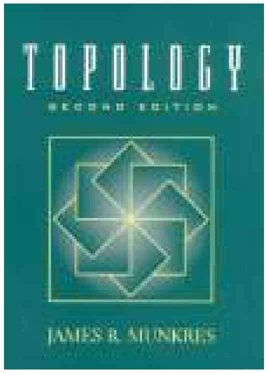
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 19. The Product Topology—Proofs of Theorems



Theorem 19.1

Theorem 19.1. Comparison of the Box and Product Topologies. The box topology on $\prod_{\alpha\in J} X_\alpha$ has as a basis all sets of the form $\prod_{\alpha\in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$. The product topology on $\prod_{\alpha\in J} X_\alpha$ has as a basis all sets of the form $\prod_{\alpha\in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$ and $U_\alpha = X_\alpha$ except for finitely many values of α.

Proof. The claim about the box topology is just a restatement of the definition of the box topology. Let $\mathcal{B}$ be the basis for the product topology that subbasis $\mathcal{S}$ generates. Then $\mathcal{B}$ consists of all finite intersections of elements of $\mathcal{S}$ by the definition of subbasis. Notice that the intersection of two elements of $\mathcal{S}_\beta$ is again an element of $\mathcal{S}_\beta$ (since $\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta)$ for any U_β, V_β open in X_β and, of course, $U_\beta \cap V_\beta$ is open in X_β), so by induction any finite intersection of elements in $\mathcal{S}_\beta$ is again in $\mathcal{S}_\beta$. So finite intersections of elements of $\mathcal{S}$ can be described as finite intersections of elements of $\mathcal{S}_\beta$ for distinct, finite number of β's.

Theorem 19.1 (continued)

Theorem 19.1. Comparison of the Box and Product Topologies. The box topology on $\prod_{\alpha\in J} X_\alpha$ has as a basis all sets of the form $\prod_{\alpha\in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$. The product topology on $\prod_{\alpha\in J} X_\alpha$ has as a basis all sets of the form $\prod_{\alpha\in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$ and $U_\alpha = X_\alpha$ except for finitely many values of α.

Proof (continued). That is, elements of $\mathcal{B}$ are of the form $B = \cap_{i=1}^n \pi_{\beta_i}^{-1}(U_{\beta_i})$ where $\beta_i \in J$, $\beta_1, \beta_2, \ldots, \beta_n$ are distinct, and U_{β_i} is open in X_{β_i}. Notice $\mathbf{x} = (x + \alpha) \in \prod_{\alpha\in J} X_\alpha$ as in $\pi_\beta^{-1}(U_\beta)$ if and only if $x_\beta \in U_\beta$. So $\pi_\beta^{-1}(U_\beta) = \prod_{\alpha\in J} Y_\alpha$ where $Y_\alpha = X_\alpha$ for $\alpha \neq \beta$ and $Y_\alpha = U_\alpha$ for $\alpha = \beta$. So basis element $B = \cap_{i=1}^n \pi_{\beta_i}^{-1}(U_{\beta_i}) = \prod_{\alpha\in J} Y_\alpha$ where $Y_\alpha = X_\alpha$ for $\alpha \notin \{\beta_1, \beta_2, \ldots, \beta_n\}$ and $Y_\alpha = U_\alpha$ for $\alpha \in \{\beta_1, \beta_2, \ldots, \beta_n\}$, as desired. □

Theorem 19.5

Theorem 19.5. Let $\{X_\alpha\}$ be an indexed family of spaces and let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$. If $\prod X_\alpha$ is given either the product or the box topology then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Proof. Let $\mathbf{x} = (x_\alpha) \in \prod \overline{A_\alpha}$. Let $\mathcal{O}$ be an open set in either the box or product topology that contains $\mathbf{x}$. Then there is $U = \prod U_\alpha$ a basis element for either th ebox or product topology that contains $\mathbf{x}$. Then $x_\alpha \in \overline{A_\alpha}$ for each $\alpha \in J$ and so there is $y_\alpha \in U_\alpha \cap A_\alpha$ by Theorem 17.5(a). Then $\mathbf{y} = (y_\alpha) \in \prod U_\alpha = U$ and so $\mathbf{y} \in U \cap \prod A_\alpha \subset \mathcal{O}$. Since $\mathcal{O}$ is an arbitrary open set containing $\mathbf{x}$ then $\mathbf{x} \in \overline{\prod A_\alpha}$ by Theorem 17.5(a) and so $\prod \overline{A_\alpha} \subset \overline{\prod A_\alpha}$.

Theorem 19.5 (continued 1)

Theorem 19.5. Let $\{X_\alpha\}$ be an indexed family of spaces and let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$. If $\prod X_\alpha$ is given either the product or the box topology then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Proof (continued). Conversely suppose $\mathbf{x} = (x_\alpha) \in \overline{\prod A_\alpha}$ (where the closure is taken in either topology). Let $\beta \in J$ and let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology so by Theorem 17.5(a) it contains a point $\mathbf{y} = (y_\alpha) \in \prod A_\alpha$. So $y_\beta \in V_\beta \cap A_\beta$. Since V_β is an arbitrary open set in X_β containing x_β then by Theorem 17.5(a) we have $x_\beta \in \overline{A_\beta}$. Since $\beta \in J$ is arbitrary then $x_\alpha \in \overline{A_\alpha}$ for all $\alpha \in J$ an d$\mathbf{x} = (x_\alpha) \in \prod \overline{A_+\alpha}$. That is, $\overline{\prod A_\alpha} \subset \prod \overline{A_\alpha}$. Hence, $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$, as claimed. □

Theorem 19.6

Theorem 19.6. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given as $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ for each $\alpha \in J$. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each functions f_α is continuous.

Proof. Let $\pi_\beta : \prod X_\alpha \to X_\beta$ be the projection mapping. The function π_β is continuous since for open $U_\beta \in X_\beta$ we have $\pi_\beta^{-1}(U_\beta) = \prod Y_\alpha$ where $Y_\alpha = X_\alpha$ for $\alpha \neq \beta$ and $Y_\alpha = U_\alpha$ if $\alpha = \beta$, and this is an element of the subbasis for the product topology and so is open. So for $f : A \to \prod X_\alpha$ continuous, the function $f_\beta = \pi_\beta \circ f$ is continuous by Theorem 18.2(c).

Theorem 19.6 (continued)

Proof (continued). Conversely, suppose each f_α is continuous for $\alpha \in J$. To prove f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in A (since each open set in $\prod X_\alpha$ is a union of finite intersections of subbasis elements and the inverse image of a union or intersection is the union or intersection of inverse images). A typical subbasis element for the product topology on $\prod X_\alpha$ is of the form $\pi_\beta^{-1}(U_\beta)$ where $\beta \in J$ and U_β is open in X_β (by Theorem 19.1 and the form of $B \in \mathcal{B}$ as given in the proof). Now $f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$ since $f_\beta = \pi_\beta \circ f$. Since f_β is hypothesized to be continuous, then $f_\beta^{-1}(U_\beta)$ is continuous and so inverse images under f of sets of the form $\pi_\beta^{-1}(U_\beta)$ are open sets. So inverse images under f of subbasis elements (and hence open set under the product topology) are open. That is, f is continuous. □