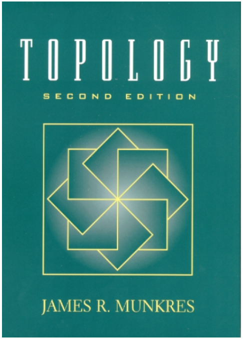
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 19. The Product Topology—Proofs of Theorems



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Theorem 19.1

Theorem 19.1. Comparison of the Box and Product Topologies. The box topology on $\prod_{\alpha \in J} X_\alpha$ has as a basis all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$. The product topology on $\prod_{\alpha \in J} X_\alpha$ has as a basis all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$ and $U_\alpha = X_\alpha$ except for finitely many values of α.

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Proof (continued). That is, elements of $\mathcal{B}$ are of the form $B = \cap_{i=1}^n \pi_{\beta_i}^{-1}(U_{\beta_i})$ where $\beta_i \in J$, $\beta_1, \beta_2, \ldots, \beta_n$ are distinct, and U_{β_i} is open in X_{β_i}. Notice $\mathbf{x} = (x + \alpha) \in \prod_{\alpha \in J} X_\alpha$ as in $\pi_\beta^{-1}(U_\beta)$ if and only if $x_\beta \in U_\beta$. So $\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in J} Y_\alpha$ where $Y_\alpha = X_\alpha$ for $\alpha \neq \beta$ and $Y_\alpha = U_\alpha$ for $\alpha = \beta$. So basis element $B = \cap_{i=1}^n \pi_{\beta_i}^{-1}(U_{\beta_i}) = \prod_{\alpha \in J} Y_\alpha$ where $Y_\alpha = X_\alpha$ for $\alpha \notin \{\beta_1, \beta_2, \ldots, \beta_n\}$ and $Y_\alpha = U_\alpha$ for $\alpha \in \{\beta_1, \beta_2, \ldots, \beta_n\}$, as desired. □

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Theorem 19.5

Theorem 19.5. Let $\{X_\alpha\}$ be an indexed family of spaces and let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$. If $\prod X_\alpha$ is given either the product or the box topology then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Proof. Let $\mathbf{x} = (x_\alpha) \in \prod \overline{A_\alpha}$. Let $\mathcal{O}$ be an open set in either the box or product topology that contains $\mathbf{x}$.

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Proof. Let $\mathbf{x} = (x_\alpha) \in \prod \overline{A_\alpha}$. Let $\mathcal{O}$ be an open set in either the box or product topology that contains $\mathbf{x}$. Then there is $U = \prod U_\alpha$ a basis element for either th ebox or product topology that contains $\mathbf{x}$. Then $x_\alpha \in \overline{A_\alpha}$ for each $\alpha \in J$ and so there is $y_\alpha \in U_\alpha \cap A_\alpha$ by Theorem 17.5(a). Then $\mathbf{y} = (y_\alpha) \in \prod U_\alpha = U$ and so $\mathbf{y} \in U \cap \prod A_\alpha \subset \mathcal{O}$.

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Proof (continued). Conversely suppose $\mathbf{x} = (x_\alpha) \in \overline{\prod A_\alpha}$ (where the closure is taken in either topology). Let $\beta \in J$ and let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology so by Theorem 17.5(a) it contains a point $\mathbf{y} = (y_\alpha) \in \prod A_\alpha$. So $y_\beta \in V_\beta \cap A_\beta$.

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Theorem 19.6

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Proof. Let $\pi_\beta : \prod X_\alpha \to X_\beta$ be the projection mapping.

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Theorem 19.6 (continued)

Proof (continued). Conversely, suppose each f_α is continuous for $\alpha \in J$. To prove f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in A (since each open set in $\prod X_\alpha$ is a union of finite intersections of subbasis elements and the inverse image of a union or intersection is the union or intersection of inverse images). A typical subbasis element for the product topology on $\prod X_\alpha$ is of the form $\pi_\beta^{-1}(U_\beta)$ where $\beta \in J$ and U_β is open in X_β (by Theorem 19.1 and the form of $B \in \mathcal{B}$ as given in the proof).

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