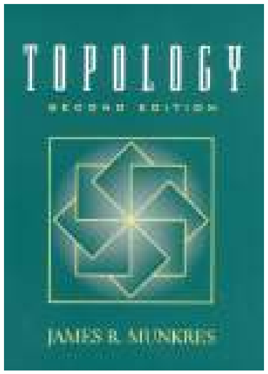
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 20. The Metric Topology—Proofs of Theorems



Theorem 20.1

Theorem 20.1. Let X be a metric space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d}(x,y) = \min\{d(x,y), 1\}$. Then $\overline{d}$ is a metric that induces the same topology as d.

Proof. "Clearly" the first two parts of the definition of metric are satisfied by $\overline{d}$. The the third part, we need to confirm the Triangle Inequality:

$$\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z).$$

We consider two cases.

Case 1. If either $d(x,y) \geq 1$ or $d(y,z) \geq 1$ then the right side of this inequality is at least 1 and so the inequality holds.

Case 2. If both $d(x,y) < 1$ and $d(y,z) < 1$ then

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \text{ by the Triangle Inequality for } d \\ &= \overline{d}(x,y) + \overline{d}(y,z). \end{aligned}$$

Since $\overline{d}(x,z) < d(x,z)$, we have the Triangle Inequality for $\overline{d}$. So $\overline{d}$ is in fact a metric.

Theorem 20.1 (continued)

Proof (continued). Let $\mathcal{B}$ be the basis for the topology induced by d and let $\mathcal{B}'$ be the basis for the topology induced by $\overline{d}$. By Lemma 13.1, the topology generated by a basis consists of all unions of basis elements. Notice that $\mathcal{B}'$ consists of all ε-balls where $\varepsilon < 1$, so $\mathcal{B}' \subset \mathcal{B}$ and the topology generated by $\mathcal{B}'$ is a subset of the topology generated by $\mathcal{B}$. But for any $B_d(x,\varepsilon) \in \mathcal{B}$ we know that $B_d(x,\varepsilon)$ can be written as a union over all elements of $B_d(x,\varepsilon)$ of balls centered at these elements of $B_d(x,\varepsilon)$ of balls centered at these elements with radius less than 1 (here we use Lemma 20.A):

$$B_x(x,\varepsilon) = \cup_{y \in B_d(x,\varepsilon)} B_d(y, \delta_y)$$

where $\delta_y = \min\{\delta, 1\}$ and $B_d(y,\delta) \subset B_d(x,\varepsilon)$ as in Lemma 20.A. So every set in the topology generated by $\mathcal{B}$ is also in the topology generated by $\mathcal{B}'$. So the topologies are the same and d and $\overline{d}$ induce the same topology on X. □

Lemma 20.2

Lemma 20.2. Let d and d' be two metrics on the set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ is and only if for such $x \in X$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$.

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $B_d(x,\varepsilon)$ be a basis element for the metric topology $\mathcal{T}$. By Lemma 13.3 (the (1)⇒(2) part) there is a basis element $\mathcal{B}' \subset B_d(x,\varepsilon)$. By Lemma 20.B, there is $B_{d'}(x,\delta) \subset B' \subset B_d(x,\varepsilon)$ and the first claim holds.

Suppose the δ/ε condition holds. Given a basis element B for the metric topology for $\mathcal{T}$ containing x, by Lemma 20.B there is a basis element $B_d(x,\varepsilon) \subset B$. By the hypothesized δ/ε condition there is $B' = B_{d'}(x,\delta) \subset B_d(x,\varepsilon) \subset B$. By Lemma 13.3 (the (2)⇒(1) part), $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Theorem 20.3

Theorem 20.3. The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = y_1, y_2, \ldots, y_n)$ be points in $\mathbb{R}^n$. Then

$$\rho(\mathbf{x},\mathbf{y}) = \max_{1\le i\le n}\{|x_i - y_i|\} \le \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{1/2} = d(\mathbf{x},\mathbf{y})$$

$$\le \left(\sum_{i=1}^n \max_{1\le i\le n} |x_i - y_i|^2\right)^{1/2} = \left(\sum_{i=1}^n \rho(\mathbf{x},\mathbf{y})^2\right)^{1/2} = \sqrt{n}\rho(\mathbf{x},\mathbf{y}). \quad (*)$$

Now for $\mathbf{y} \in B_d(\mathbf{x},\varepsilon)$ we have $d(\mathbf{x},\mathbf{y}) < \varepsilon$ and so $\rho(\mathbf{x},\varepsilon) < d(\mathbf{x},\mathbf{y}) < \varepsilon$ (by $(*)$) and so $\mathbf{y} \in B_\rho(\mathbf{x},\varepsilon)$. Therefore $B_d(\mathbf{x},\varepsilon) \subset B_\rho(\mathbf{x},\varepsilon)$ and by Lemma 20.2, the metric topology induced by d is finer than the metric topology induced by ρ. For $\mathbf{y} \in B_\rho(\mathbf{x},\varepsilon/\sqrt{n})$ we have $\rho(\mathbf{x},\mathbf{y} < \varepsilon/\sqrt{n}$ and so $d(\mathbf{x},\mathbf{y}) \le \sqrt{n}\rho(\mathbf{x},\mathbf{y}) = \sqrt{n}(\varepsilon/\sqrt{n}) = \varepsilon$ (by $(*)$) and $\mathbf{y} \in B_d(\mathbf{x},\varepsilon)$.

Theorem 20.3 (continued 1)

Proof (continued). Therefore $B_\rho(\mathbf{x},\varepsilon) \subset B_d(\mathbf{x},\varepsilon)$ and by Lemma 20.2, the metric topology induced by ρ is finer than the metric topology induced by d. Hence, the metric topologies under d and ρ are the same.
Now to show that the product topology is the same as the metric topology induced by ρ. First, let $B = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ be a basis element for the product topology. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in B$. Then $x_i \in (a_i, b_i)$ for each i and so there is $\varepsilon_i > 0$ such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i)$ (take, for example, $\varepsilon_i = \min\{x_i - a_i, b_i - x_i\}$). Set $\varepsilon = \min_{1\le i\le n}\{\varepsilon_i\}$. Then

$$B_\rho(x,\varepsilon) \subset (x_1-\varepsilon_1, x_1+\varepsilon_1)\times(x_2-\varepsilon_2, x_2+\varepsilon_2)\times\cdots\times(x_n-\varepsilon_n, x_n+\varepsilon_n) \subset B.$$

So by Lemma 13.3 (the (2)⇒(1) part), the metric topology induced by ρ is finer than the product topology.

Theorem 20.3 (continued 2)

Theorem 20.3. The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof (continued). Conversely, let $B_\rho(\mathbf{x},\varepsilon)$ be a basis element for the metric topology induced by ρ. Let $\mathbf{y} \in B_\rho(\mathbf{x},\varepsilon)$. Let

$$B = B_\rho(\mathbf{x},\varepsilon) = (x_1-\varepsilon_1, x_1+\varepsilon_1)\times(x_2-\varepsilon_2, x_2+\varepsilon_2)\times\cdots\times(x_n-\varepsilon_n, x_n+\varepsilon_n).$$

Then $B \subset B_\rho(\mathbf{x},\varepsilon)$ and B is a basis element for the product topology. So by Lemma 13.3 (the (2)⇒(1) part), the product topology is finer than the metric topology induced by ρ. Therefore the box topology and the metric topology induced by ρ (AND the metric topology induced by d, as shown above) are the same. □

Theorem 20.4

Theorem 20.4. The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. These three topologies are all different if J is infinite.

Proof. Let $\mathbf{x} = (x_\alpha)_{\alpha\in J} \in \mathbb{R}^J$ and let B be a basis element for the product topology which contains $\mathbf{x}$. Then by Theorem 19.1, $B = \prod U_\alpha$ where each U_α is open in $\mathbb{R}$ and $U_\alpha = \mathbb{R}$ for all but finitely many values of α (say $\alpha_1, \alpha_2, \ldots, \alpha_n$). For each $i = 1, 2, \ldots, n$, choose $\varepsilon_i > 0$ so that $B_{\overline{d}}(x_i, \varepsilon_i) \subset U_{\alpha_i}$ (which can be done since U_α is open in $\mathbb{R}$ under the standard topology and d and $\overline{D}$ induce the same topology on $\mathbb{R}$ by Theorem 20.1). Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. Then $B_{\overline{\rho}}(\mathbf{x},\varepsilon) \subset \prod U_\alpha$. Since $B_{\overline{\rho}}(\mathbf{x},\varepsilon)$ is a basis element for the metric topology on $\mathbb{R}^J$ induced by metric $\overline{\rho}$ (i.e., the uniform topology). By Lemma 13.3 (the (2)⇒(1) part), the uniform topology is finer than the product topology.

Theorem 20.4 (continued)

Theorem 20.4. The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. These three topologies are all different if J is infinite.

Proof (continued). Now let $B = B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$ be a basis element for the uniform topology. Then the open set

$$U = \prod_{\alpha \in J} (x_\alpha - \varepsilon/2, x_\alpha + \varepsilon/2)$$

is a basis element for the box topology and $\mathbf{x} \in U \subset B$. So by Lemma 13.3, the box topology is finer than the uniform topology.

The fact that the three topologies are different when J is infinite is left as a homework exercise. □

Theorem 20.5

Theorem 20.5. Let $\overline{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\mathbf{x}$ and $\mathbf{y}$ are two points in $\mathbb{R}^\omega = \mathbb{R}^{\mathbb{N}}$, define

$$D(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{N}} \left\{ \frac{\overline{d}(x_i, y_i)}{i} \right\}.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$. That is, $\mathbb{R}^\omega$ under the product topology is metrizable.

Proof. The first two parts of the definition of metric are clearly satisfied by D. Notice that for all $i \in \mathbb{N}$, by the Triangle Inequality for $\overline{d}$,

$$\frac{\overline{d}(x_i, z_i)}{i} \le \frac{\overline{d}(x_i, y_i)}{i} + \frac{\overline{d}(y_i, z_i)}{i} \le D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}).$$

So

$$D(\mathbf{x}, \mathbf{z}) = \sup_{i \in \mathbb{N}} \left\{ \frac{\overline{d}(x_i, z_i)}{i} \right\} \le D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}),$$

and the Triangle Inequality holds for D and D is a metric on $\mathbb{R}^\omega$.

Theorem 20.5 (continued 1)

Proof (continued). Let U be an open set in the metric topology induced by D and let $\mathbf{x} \in U$. Choose $\varepsilon > 0$ such that $B_D(\mathbf{x}, \varepsilon) \subset U$ and choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Let V be the basis element for the product topology

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots.$$

Then for any $\mathbf{y} \in \mathbb{R}^\omega$ we have

$$\frac{\overline{d}(x_i, y_i)}{i} = \frac{\min\{|x_i - y_i|, 1\}}{i} \le \frac{1}{i} \le \frac{1}{N} \text{ for } i \ge N.$$

Therefore

$$D(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{N}} \left\{ \frac{\overline{d}(x_i, y_i)}{i} \right\} \le \max \left\{ \frac{\overline{d}(x_1, y_1)}{1}, \frac{\overline{d}(x_2, y_2)}{2}, \ldots, \frac{\overline{d}(x_N, y_N)}{N}, \frac{1}{N} \right\}.$$

If $\mathbf{y} \in V$ then this maximum is less than ε and so $V \subset B_D(\mathbf{x}, \varepsilon) \subset U$. So for every open set U in the metric topology induced by D, there is a basis element V of the product topology such that $V \subset U$.

Theorem 20.5 (continued 2)

Proof (continued). So by Lemma 13.3 (the (2)⇒(1) part), the product topology is finer than the metric topology.
Conversely, let U be a basis element for the product topology. Then by Theorem 19.1, $U = \prod_{i \in \mathbb{N}} U_i$ where each U_i is open in $\mathbb{R}$ for all $i \in \mathbb{N}$ and $U_i = \mathbb{R}$ for all i except for finitely many, say $i = \alpha_1, \alpha_2, \ldots, \alpha_n$. Let $\mathbf{x} \in U$. Choose ε_i, where $0 < \varepsilon_i \le 1$, such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i$ for $i = \alpha_1, \alpha_2, \ldots, \alpha_n$. Define $\varepsilon = \min\{\varepsilon_i / i \mid i = \alpha_1 . \alpha_2, \ldots, \alpha_n\}$. Let $\mathbf{Y} \in B_D(\mathbf{x}, \varepsilon)$. Then for all $i \in \mathbb{N}$,

$$\frac{\overline{d}(x_i, y_i)}{i} \le \sup_{i \in \mathbb{N}} \left\{ \frac{\overline{d}(x_i, y_i)}{i} \right\} = D(\mathbf{x}, \mathbf{y}) < \varepsilon.$$

If $i = \alpha_1, \alpha_2, \ldots, \alpha_n$ then $\varepsilon \le \varepsilon_i / i$ (by the definition of ε), so that $\overline{d}(x_i, y_i) < \varepsilon_i \le 1$ (by the choice of ε_i). Since $\overline{d}(x_i, y_i) = \min\{|x_i - y_i|, 1\} < 1$, it must be that $\overline{d}(x_i, y_i) = |x_i - y_i|$ and so $|x_i - y_i| < \varepsilon_i$ for $i = \alpha_1, \alpha_2, \ldots, \alpha_n$. Therefore $\mathbf{y} \in \prod_{i \in \mathbb{N}} U_i = U$ and so $B_D(\mathbf{x}, \varepsilon) \subset U$.

Theorem 20.5 (continued 3)

Theorem 20.5. Let $\overline{d}(a, b) = \min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\mathbf{x}$ and $\mathbf{y}$ are two points in $\mathbb{R}^\omega = \mathbb{R}^\mathbb{N}$, define

$$D(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{N}} \left\{ \frac{\overline{d}(x_i, y_i)}{i} \right\}.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$. That is, $\mathbb{R}^\omega$ under the product topology is metrizable.

Proof. So for every basis element U of the product topology, there is a basis element $V = B_D(\mathbf{x}, \varepsilon)$ of the metric topology such that $V \subset U$. So by Lemma 13.3 (the (2)$\Rightarrow$(1) part), the metric topology is finer than the product topology. Hence, the metric topology and the product topology are the same. That is, the metric D induces the product topology on $\mathbb{R}^\omega$. □