

Theorem 21.1

Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 21. The Metric Topology (Continued)—Proofs of Theorems



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Theorem 21.1.

Theorem 21.1 (continued)

Theorem 21.1. Let $f : X \rightarrow Y$. Let X and Y be metrizable with metrics d_X and d_Y , respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Proof (continued). Conversely suppose that the ε/δ condition is satisfied. Let V be an open set in Y . Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since V is open an $df(x) \in V$ then by Lemma 20.B there is $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset V$. By the ε/δ hypothesis, there is $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$ (i.e., $f(y) \in B(f(x), \varepsilon)$). So $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. So $x \in B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \subset V$. Therefore, by Lemma 20.B, $f^{-1}(V)$ is open and so f is continuous. \square

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$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Proof. Suppose f is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Consider the set $f^{-1}(B(f(x), \varepsilon))$. Since f is continuous then by definition of continuity, $f^{-1}(B(f(x), \varepsilon))$ is open in X since $B(f(x), \varepsilon)$ is open and $x \in f^{-1}(B(f(x), \varepsilon))$ then by Lemma 20.A there is $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. Then $d_X(x, y) < \delta$ implies $y \in B(f(x), \delta)$, so $f(y) \in B(f(x), \varepsilon)$ and $d_Y(f(x), f(y)) < \varepsilon$.

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Lemma 21.2. The Sequence Lemma

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Let X be a topological space. Let $A \subset X$. If there is a sequence of points of A converging to x , then $x \in \bar{A}$. If X is metrizable and $x \in \bar{A}$ then there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \rightarrow x$.

Proof. Suppose that $\{x_n\} \rightarrow x$ where $\{x_n\} \subset A$. Then any given neighborhood U of x , there is, by the definition of limit of a sequence (see Section 17) $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. So every neighborhood of x contains an element of the sequence and hence an element of set A . Then by Theorem 17.5 (part (a)), $x \in \bar{A}$.

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Lemma 21.2 (continued)

Lemma 21.2. The Sequence Lemma.

Let X be a topological space. Let $A \subset X$. If there is a sequence of points of A converging to x , then $x \in \overline{A}$. If X is metrizable and $x \in \overline{A}$ then there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \rightarrow x$.

Proof (continued). Conversely, suppose that X is metrizable and $x \in \overline{A}$. Let d be a metric for the topology of X . For each $n \in \mathbb{N}$, consider

$B_d(x, 1/n)$. This is an open set containing x and so by Theorem 17.5

(part (a)), $B(x, 1/n)$ contains an element of A , say x_n . Then consider the

resulting sequence $\{x_n\}$. Any open set U which contains x contains, for

some $\varepsilon > 0$, $B_d(x, \varepsilon) \subset U$, by Lemma 20.B. Since $\varepsilon > 0$, then

for some $N \in \mathbb{N}$ we have $1/N < \varepsilon$. So by construction of the sequence, for

all $n \geq N$ we have $s_n \in B(x, 1/n) \subset B(x, 1/N) \subset B(x, \varepsilon) \subset U$. So

$\{x_n\} \rightarrow x$ by the definition of convergent sequence. \square

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Theorem 21.3

Theorem 21.3 (continued)

Theorem 21.3. Let $f : X \rightarrow Y$. If f is continuous then for every

convergent sequence $\{x_n\} \rightarrow x$ in X , the sequence $\{f(x_n)\} \rightarrow f(x)$ in Y .

If X is metrizable and for any sequence $\{x_n\} \rightarrow x$ in X we have

$\{f(x_n)\} \rightarrow f(x)$ in Y , then f is continuous.

Proof (continued). Conversely, suppose X is metrizable and suppose for any $x \in X$ and any sequence $\{x_n\} \rightarrow x$ in X we have $\{f(x_n)\} \rightarrow f(x)$. Let $A \subset X$. If $x \in \overline{A}$ then there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \rightarrow x$ by Lemma 21.2 (part 2). By hypothesis, $\{f(x_n)\} \rightarrow f(x)$. Since $\{x_n\} \subset A$ then $f(x_n) \in \overline{f(A)}$ by Lemma 21.2 (part 1; notice that this does not require the metrizability of Y). Since $x \in \overline{A}$ is arbitrary, then $f(\overline{A}) \subset \overline{f(A)}$. Hence, by Theorem 18.1 (the (2) \Rightarrow (1) part), f is continuous. \square

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convergent sequence $\{x_n\} \rightarrow x$ in X , the sequence $\{f(x_n)\} \rightarrow f(x)$ in Y .

If X is metrizable and for any sequence $\{x_n\} \rightarrow x$ in X we have

$\{f(x_n)\} \rightarrow f(x)$ in Y , then f is continuous.

Proof. Suppose f is continuous and let $\{x_n\} \rightarrow x$ in X . Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open and contains x . Since $\{x_n\} \rightarrow x$, by the definition of convergent sequence (see Section 17), there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in f^{-1}(V)$. Then $f(x_n) \in V$ for all $n \geq N$ and so (by definition again) $\{f(x_n)\} \rightarrow f(x)$.

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Theorem 21.5

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Theorem 21.5. If X is a topological space and if $f, g : X \rightarrow \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all $x \in X$ then f/g is continuous.

Proof. The map $h : X \rightarrow \mathbb{R} \times \mathbb{R}$ defined by $h(x) = (f(x), g(x))$ is continuous by Theorem 18.4 (“Maps Into Products”). The function $f + g$ equals the composition of h and the addition operation $+$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Therefore $f + g$ is continuous by Theorem 18.2 part (c).

Similarly, $f - g$ is the composition of h and the subtraction operation $-$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $f \cdot g$ is the composition of h and the multiplication operation \cdot : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, and f/g is the composition of h and the division operation \div : $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}$. So each of these is also continuous. \square

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Theorem 21.6

Theorem 21.6. Uniform Limit Theorem.

Let $f_n : X \rightarrow Y$ be a sequence of continuous functions from the topological space X to the metric space Y . If $\{f_n\}$ converges uniformly to f , then f is continuous.

Proof. Let V be open in Y and let $x_0 \in f^{-1}(V)$. Let $y_0 = f(x_0) \in V$ and choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subset V$ (by Lemma 20.B). Since $\{f_n\}$ converges uniformly to f on X then there is $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in X$ we have

$$(f_n(x), f(x)) < \varepsilon/3. \quad (*)$$

Since f_N is continuous, there is a neighborhood U of x_0 such that

$$f_N(U) \subset B(f_N(x_0), \varepsilon/3) \quad (**)$$

by Theorem 18.1 (the (1) \Rightarrow (4) part where $B(f_N(x_0), \varepsilon/3)$ is treated as a neighborhood of $f(x_0)$).

Theorem 21.6 (continued)

Proof (continued). Next, if $x \in U$ then

$$d(f(x), f_N(x)) < \varepsilon/3 \text{ by } (*) \text{ with } n = N$$

$$d(f_N(x), f_N(x_0)) < \varepsilon/3 \text{ by } (**) \text{ since } x \in U$$

$$d(f_N(x_0), f(x_0)) < \varepsilon/3 \text{ by } (*) \text{ with } n = N \text{ and } x = x_0.$$

Then by the Triangle Inequality,

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $x \in U$. So U is a neighborhood of x_0 with $f(U) \subset B(f(x_0), \varepsilon) \subset V$. So by Theorem 18.1 (the (4) \Rightarrow (1) part), f is continuous. \square