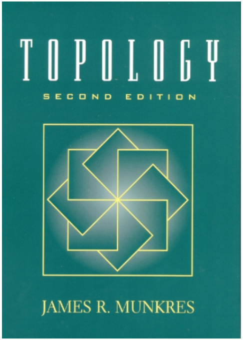
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 21. The Metric Topology (Continued)—Proofs of Theorems



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$$d_X(x, y) < \delta \;\Rightarrow\; d_Y(f(x), f(y)) < \varepsilon.$$

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Proof (continued). Conversely suppose that the ε/δ condition is satisfied. Let V be an open set in Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$.

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Proof (continued). Conversely suppose that the ε/δ condition is satisfied. Let V be an open set in Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since V is open an d$f(x) \in V$ then by Lemma 20.B there is $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset V$. By the ε/δ hypothesis, there is $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$ (i.e., $f(y) \in B(f(x), \varepsilon)$).

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Lemma 21.2

Lemma 21.2. The Sequence Lemma.
Let X be a topological space. Let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. If X is metrizable and $x \in \overline{A}$ then there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \to x$.

Proof. Suppose that $\{x_n\} \to x$ where $\{x_n\} \subset A$. Then any given neighborhood U of x, there is, by the definition of limit of a sequence (see Section 17) $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

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Proof (continued). Conversely, suppose that X is metrizable and $x \in \overline{A}$. Let d be a metric for the topology of X. For each $n \in \mathbb{N}$, consider $B_d(x, 1/n)$. This is an open set containing x and so by Theorem 17.5 (part (a)), $B(x, 1/n)$ contains an element of A, say x_n. Then consider the resulting sequence $\{x_n\}$.

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Theorem 21.3. Let $f : X \to Y$. If f is continuous then for every convergent sequence $\{x_n\} \to x$ in X, the sequence $\{f(x_n)\} \to f(x)$ in Y. If X is metrizable and for any sequence $\{x_n\} \to x$ in X we have $\{f(x_n)\} \to f(x)$ in Y, then f is continuous.

Proof. Suppose f is continuous and let $\{x_n\} \to x$ in X. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open and contains x.

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Theorem 21.5

Theorem 21.5. If X is a topological space and if $f, g : X \to \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all $x \in X$ then f/g is continuous.

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = (f(x), g(x))$ is continuous by Theorem 18.4 ("Maps Into Products"). The function $f + g$ equals the composition of h and the addition operation $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Therefore $f + g$ is continuous by Theorem 18.2 part (c).

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Similarly, $f - g$ is the composition of h and the subtraction operation $- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f \cdot g$ is the composition of h and the multiplication operation $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and f/g is the composition of h and the division operation $\div : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$. So each of these is also continuous. □

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Theorem 21.6

Theorem 21.6. Uniform Limit Theorem.
Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $\{f_n\}$ converges uniformly to f, then f is continuous.

Proof. Let V be open in Y and let $x_0 \in f^{-1}(V)$.

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$$(f_n(x), f(x)) < \varepsilon/3. \qquad (*)$$

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$$(f_n(x), f(x)) < \varepsilon/3. \qquad (*)$$

Since f_N is continuous, there is a neighborhood U of x_0 such that

$$f_N(U) \subset B(f_N(x_0), \varepsilon/3) \qquad (**)$$

by Theorem 18.1 (the (1)⇒(4) part where $B(f_N(x_0), \varepsilon/3)$ is treated as a neighborhood of $f(x_0)$).

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Since f_N is continuous, there is a neighborhood U of x_0 such that

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by Theorem 18.1 (the (1)⇒(4) part where $B(f_N(x_0), \varepsilon/3)$ is treated as a neighborhood of $f(x_0)$).

Theorem 21.6 (continued)

Proof (continued). Next, if $x \in U$ then

$$d(f(x), f_N(x)) < \varepsilon/3 \text{ by } (*) \text{ with } n = N$$

$$d(f_N(x), f_N(x_0)) < \varepsilon/3 \text{ by } (**) \text{ since } x \in U$$

$$d(f_N(x_0), f(x_0)) < \varepsilon/3 \text{ by } (*) \text{ with } n = N \text{ and } x = x_0.$$

Then by the Triangle Inequality,

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $x \in U$.

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$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $x \in U$. So U is a neighborhood of x_0 with $f(U) \subset B(f(x_0), \varepsilon) \subset V$. So by Theorem 18.1 (the (4)⇒(1) part), f is continuous. □

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$$d(f_N(x_0), f(x_0)) < \varepsilon/3 \text{ by } (*) \text{ with } n = N \text{ and } x = x_0.$$

Then by the Triangle Inequality,

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $x \in U$. So U is a neighborhood of x_0 with $f(U) \subset B(f(x_0), \varepsilon) \subset V$. So by Theorem 18.1 (the (4)⇒(1) part), f is continuous. □