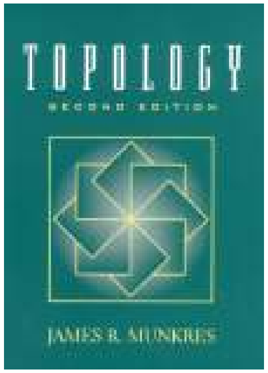
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions
Section 22. The Quotient Topology—Proofs of Theorems



Lemma 22.A

Lemma 22.A. Let X and Y be topological spaces. Then $p : X \to Y$ is a quotient map if and only if p is continuous and maps saturated open sets of X to open sets of Y.

Proof. Suppose p is a quotient map. Then p is continuous (since the inverse image of every open set in Y has an open inverse image in X, by definition of quotient map). Also, for any open saturated set $U \subset X$, there is open $A \subset Y$ with $p^{-1}(A) = U$. Since $U = p(A)$ is open in Y then (by definition of quotient map) A is open in X. So if p is a quotient map then p is continuous and maps saturated open sets of X to open sets of Y (and similarly, saturated closed sets of X to closed sets of Y).

Lemma 22.A (continued)

Lemma 22.A. Let X and Y be topological spaces. Then $p : X \to Y$ is a quotient map if and only if p is continuous and maps saturated open sets of X to open sets of Y.

Proof (continued). Now suppose p is continuous and maps saturated open sets of X to open sets of Y. Since p is continuous, then for any open $U \subset Y$ we have $p^{-1}(U)$ open in X. Now suppose $p^{-1}(U)$ is open in X. Then, by the not above, $p^{-1}(U)$ is saturated since it is the inverse image of some set in Y (namely, U). Since $p^{-1}(U)$ is a saturated open set, we have hypothesized that $p(p^{-1}(U)) = U$ is open in Y. So U is open in X if and only if $p^{-1}(U)$ is open in X. That is, p is a quotient map. □

Theorem 22.1

Theorem 22.1. Let $p : X \to Y$ be a quotient map. Let A be a subspace of X that is saturated with respect to p. Let $q : A \to p(A)$ be the map obtained by restricting p to S, $q = p|_A$.

(1) If A is either open or closed in X, then a is a quotient map.
(2) If p is either an open or a closed map, then q is a quotient map.

Proof. STEP 1. Let $V \subset p(A)$. Then for each $v \in V$ there must be $a \in A$ such that $p(a) = v$. So $p^{-1}(\{v\}) \cap A$ includes a and so is nonempty. Since A is saturated with respect to p, then $p^{-1}(V) \subset A$. Since $q = p|_A$ then $q^{-1}(V)$ is all the points of A mapped by p into V. That is,

$$\text{if } V \subset p(A) \text{ then } q^{-1}(V) = p^{-1}(V).$$

For any subsets $U \subset X$ and $A \subset X$ we have $p(U \cap A) \subset p(U) \cap p(A)$ since $U \cap A \subset U$ and $U \cap A \subset A$. Suppose $y = p(u) = p(a) \in p(U) \cap p(A)$ for $u \in U$ and $a \in A$. Since A is saturated with respect to p and $p^{-1}(p(a))$ includes $a \in A$ (and so $p^{-1}(p(a)) \cap A \neq \varnothing$), then $p^{-1}p(a) \subset A$.

Theorem 22.1 (continued 1)

Proof (continued). Since $u \in p^{-1}(p(a)) \subset A$ then $x \in U \cap A$. So $y = p(u) \in p(U \cap A)$. So $p(U) \cap p(A) \subset p(U \cap A)$. Therefore

$$\text{if } U \subset X \text{ then } p(U \cap A) = p(U) \cap p(A).$$

STEP 2. Suppose set A is open in X. Let $V \subset p(A)$ where $q^{-1}(V)$ is open in A. Since $q^{-1}(V)$ is open in A and A is open in X, then $q^{-1}(V)$ is open in X. Since $q^{-1}(V) = p^{-1}(V)$ by Step 1, then $p^{-1}(V)$ is open in X. Since p is a quotient map then (by definition) V is open in Y. So V is open in $p(A)$. So if $q^{-1}(V)$ is open in A then V is open in $p(A)$ (recall $q : A \to p(A)$). Since p is a quotient map, then it is continuous (inverse images of open sets are open) and q is a restriction of p, then q is continuous (restrictions of continuous functions are continuous by Theorem 18.2(d)). So inverse images of open sets are open under q. Therefore, $q : A \to p(A)$ and V is open in $p(A)$ if and only if $q^{-1}(V)$ is open in A.

Theorem 22.1 (continued 2)

Proof (continued). Since $p : X \to Y$ is surjective (onto) and $q = p|_A$, then $q : A \to p(A)$ is surjective. That is, q is a quotient map, and (1) follows for A open.

Suppose map p is open. Let $A \subset p(A)$ where $q^{-1}(V)$ is open in A. Since $p^{-1}(V) = q^{-1}(V)$ by Step 1, then $p^{-1}(V)$ is open in A. That is, $p^{-1}(V) = A \cap U$ for some open set U in X. Now $p(p^{-1}(V)) = V$ because p is onto (surjective). Then $V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$ by Step 1. Since p is a quotient map and U is open in X then $p(U)$ is open in Y. Hence V is open in $p(A)$. As in the previous paragraph, this is sufficient to show that q is a quotient map and (2) follows for p an open map.

STEP 3. The arguments in Step 2 follow through with "open" replace with "closed." Therefore, (1) follows for set A closed and (2) follows for map p closed. $\square$

Theorem 22.2

Theorem 22.2. Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous. f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Theorem 22.2 (continued 1)

Proof. For each $y \in Y$, the set $g(p^{-1}(\{y\}))$ is a one-point set in Z since g is constant on $p^{-1}(\{y\})$. Define $f(y)$ to be this one point. Then $f : Y \to Z$ and for each $x \in W$ we have $f(p(x)) = g(x)$. So function f exists as claimed.

If f is continuous, then the composition $g = f \circ p$ is continuous (since p is a quotient map and so by definition is continuous).

Suppose g is continuous. Let V be an open set in Z. Then $g^{-1}(V)$ is open in X. But $g^{-1}(V) = p^{-1}(f^{-1}(V))$ by above. Since p is a quotient map, $p^{-1}(f^{-1}(V))$ is open if and only if $f^{-1}(V)$ is open and hence, since $p^{-1}(f^{-1}(V))$ *is* open, then $f^{-1}(V)$ is open and so f is continuous. So f is continuous if and only if g is continuous.

Suppose f is a quotient map. Then g is the composite of two quotient maps and hence is a quotient map (see page 141 for details).

Theorem 22.2 (continued 2)

Proof (continued). Suppose that g is a quotient map. Then, by the definition of quotient map, g is onto (surjective). Therefore f is surjective. Let $V \subset Z$ and suppose $f^{-1}(V)$ is open in Y. Then $p^{-1}(f^{-1}(V))$ is open in X because p is continuous. Since $g^{-1}(V) = p^{-1}(f^{-1}(V))$, then $g^{-1}(V)$ is open. Since g is a quotient map, then V is open in Z. So if $f^{-1}(V)$ is open then V is open. We have assumed that f is a quotient map, so g is continuous and by above, f is continuous. So if V is open in Z then $f^{-1}(V)$ is open in Y. Therefore, f is a quotient map. □

Corollary 22.3

Corollary 22.3. Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X: $X^* = \{g^{-1}\{z\}) \mid z \in Z\}$. Let X^* have the quotient topology.

(a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

(b) If Z is Hausdorff, so is X^*.

Corollary 22.3 (continued 1)

Proof. Let $p : X \to X^*$ be the projection map that carries each point in X to the element of X^* containing it. By Theorem 22.2, since g is hypothesized to be continuous, g induces a continuous map $f : X^* \to Z$. As argued in the proof of Theorem 22.2, since $f \circ p = g$ and g is surjective, then f is surjective. Suppose $g^{-1}(\{z_1\}) = g^{-1}(\{z_2\})$. Let $x_1, x_2 \in X$ such that $p(x_1) = g^{-1}(\{z_1\}))$ and $p(x_2) = g^{-1}(\{z_2\})$ (notice that projection p is onto X^*). So $x_1 \in g^{-1}(\{z_1\})$ and $g^{-1}(\{z_2\})$ must be disjoint (the $g^{-1}(\{z\})$'s partition X). Hence $z_1 \neq z_2$ and $x_1 \neq x_2$ and so $g(x_1) = z_1 \neq z_2 = g(x_2)$. So $(f \circ p)(x_1) = f(g^{-1}(\{z_1\})) = g(x_1) = z_1$ and $(f \circ p)(x_2) = f(g^{-1}(\{z_2\}) = g(x_2) = z_2$. That is, $f(g^{-1}(\{z_1\})) \neq f(g^{-1}(\{z_2\}))$, and so f is one to one. So f is a bijection.

Corollary 22.3 (continued 2)

Proof (continued). Suppose f is a homeomorphism. Then f maps open sets to open sets and since f is continuous, inverse images of pen sets are open. So f is a quotient map. Now p is a quotient map by definition (see the definition of "quotient topology"). So the composition $g = f \circ p$ is a quotient map. Then by Theorem 22.2, f is a quotient map. Since f is bijective as argued above, then f is a homeomorphism. So (a) follows.

Suppose Z is Hausdorff. For distinct elements of X^*, their images under f are distinct since f is one to one by (a). So in Z these images have disjoint neighborhoods U and V. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint (f is a bijection) and open (f is continuous by (a)) and are neighborhoods of the two given points of X^*. Hence X^* is Hausdorff. □