Lemma 23.2. Let $X$ be a subspace of $X$. Two disjoint nonempty sets $A$ and $B$ whose union is $X$ and neither contains a limit point of $X$ are open and closed in $X$. The closure of $A$ in $X$ is $A$, and its interior in $X$ is $A$. Since $A$ is open in $X$, $A$ is a separation of $X$. Therefore, $A$ is both closed and open in $X$. This means that $A$ is a separation of $X$. Since $A$ and $B$ are both closed and open in $X$, $A$ and $B$ form a separation of $X$. Therefore, $X$ is connected.

Section 23. Connected Spaces—Proofs of Theorems

Chapter 3. Connectedness and Compactness

Lemma 23.1. Let $X$ be a subspace of $X$. Two disjoint nonempty sets $A$ and $B$ whose union is $X$ and neither contains a limit point of $X$ are open and closed in $X$. Therefore, $X$ is connected.
Theorem 2.3.6. A finite Cartesian product of connected spaces is connected.

**Proof.** We prove the result for two connected spaces $X$ and $Y$ and then.

Theorem 2.3.5. The image of a connected space under a continuous map is connected.

**Proof.** Let $f: X \to Y$ be a continuous function where $X$ is connected.

Theorem 2.3.4. Let $A$ be a connected subspace of $X$. If $A \cap B \neq \emptyset$, then $B$ is also connected.

**Proof.** Let $A \subseteq B$. Then $A$ is a connected subspace of $X$. If $A \subseteq B$, then $B$ is also connected.

Theorem 2.3.3. The union of a collection of connected subspaces of $X$ is connected.

**Proof.** Let $A_0 \subseteq A \subseteq \bigcup A_i$ be a collection of connected subspaces of $X$ and $D$.

Theorem 2.3.2. Let $Y = \bigcup A_i$ and assume $Y = \bigcup A_i$ where $A_0 \subseteq A \subseteq \bigcup A_i$. Hence $A_0 \subseteq A$.

Theorem 2.3.1. The union of a collection of connected subspaces of $X$ is connected.

**Proof.** Let $A_0 \subseteq A \subseteq \bigcup A_i$ and assume $A_0 \subseteq A \subseteq \bigcup A_i$. Hence $A_0 \subseteq A$.

Theorem 2.2.4. Let $A$ and $B$ be a connected subspace of $X$. If $A \cap B \neq \emptyset$, then $B$ is also connected.

**Proof.** Let $A \subseteq B$. Then $A$ is a connected subspace of $X$. If $A \subseteq B$, then $B$ is also connected.

Theorem 2.2.3. The union of a collection of connected subspaces of $X$ is connected.

**Proof.** Let $A_0 \subseteq A \subseteq \bigcup A_i$ be a collection of connected subspaces of $X$ and $D$.