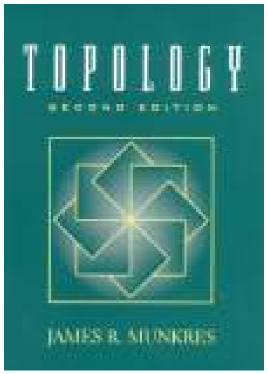
Introduction to Topology

Chapter 3. Connectedness and Compactness
Section 23. Connected Spaces—Proofs of Theorems



Lemma 23.1

Lemma 23.1. Let Y be a subspace of X. Two disjoint nonempty sets A and B whose union is Y form a separation of Y if and only if A contains no limit points of B and B contains not limit points of A.

Proof. Suppose that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is $\overline{A} \cap Y$ (where $\overline{A}$ denotes the closure of A in X) by Theorem 17.4. Since A is closed in Y, $Z = \overline{A} \cap Y$. Since $A \cap B = \varnothing$ then $\overline{A} \cap B = \varnothing$. Since $\overline{A}$ is the union of A and its limit points by Theorem 17.6, then B contains no limit points of A. Similarly, A contains no limit points of B. So a separation of Y is a pair of nonempty sets A and B whose union is Y and neither contain a limit point of the other.

Lemma 23.1 (continued)

Lemma 23.1. Let Y be a subspace of X. Two disjoint nonempty sets A and B whose union is Y form a separation of Y if and only if A contains no limit points of B and B contains not limit points of A.

Proof (continued). Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Then $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$ (again, since the closure of a set is the union of the set and its limit points by Theorem 17.6). Since $A \cup B = Y$, then $(\overline{A} \cap Y) \cup (\overline{B} \cap Y) = Y$. Since $A \cap (\overline{B} \cap Y) = \varnothing$ then we must have $\overline{A} \cap Y \subset A$ and similarly $\overline{B} \cap Y \subset B$. But $A \subset \overline{A} \cap Y$ and $B \subset \overline{B} \cap Y$ also, so $A = \overline{A} \cap Y$ and $B = \overline{B} \cap Y$. Thus A and B are both closed in Y and so $A = T \setminus B$ and $B = Y \setminus A$ are both open in Y. That is, A and B is a separation of Y. □

Lemma 23.2

Lemma 23.2. If sets C and D form a separation of X and if Y is a connected subspace of X, then Y lies entirely in either C or in D.

Proof. Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y. These two sets are disjoint and their union in Y. ASSUME both are nonempty. Then these two sets form a separation of Y, CONTRADICTING the hypothesis that Y is connected. So either $C \cap Y$ or $D \cap Y$ is an empty set and so Y lies entirely in either X or in D. □

Theorem 23.3

Theorem 23.3. The union of a collection of connected subspaces of X that have a point in common is connected.

Proof. Let $\{A_\alpha\}$ be a collection of subspaces of a space X. Let p be a point in $\cap A_\alpha$. Let $Y = \cup A_\alpha$ and ASSUME $Y = C \cup D$ where C and D are a separation of Y. Point p must be in either C or D. WLOG, say $p \in C$. Since A_α is connected, it must lie entirely in either C or in D, by Lemma 23.2. It cannot lie in D since $p \in A_\alpha$ and $p \in C$. Hence $A_\alpha \subset C$ for every α. So $Y = \cup A_\alpha \subset C$. But the CONTRADICTS the fact that D is nonempty. So the assumption that there is a separation of Y is false and hence the union $Y = \cup A_\alpha$ is connected. □

Theorem 23.4

Theorem 23.4. Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Proof. Let A be connected and let $A \subset B \subset \overline{A}$. ASSUME that $B = C \cup D$ where C and D are a separation of B. By Lemma 23.2, the set A must lie entirely in C or in D. WLOG, suppose $A \subset C$. Then $\overline{A} \subset \overline{C}$. Since $\overline{C}$ and D are disjoint by Lemma 23.1, then $B \cap D = \varnothing$. But this CONTRADICTS the fact that as part of a separation, D is a nonempty subset of B. So the assumption that a separation of B exists is false, and so B is connected. □

Theorem 23.5

Theorem 23.5. The image of a connected space under a continuous map is connected.

Proof. Let $F : X \to Y$ be a continuous function where X is connected. Let $Z = f(X)$. Since the map obtained from f by restricting its range to the space Z is also continuous (by Theorem 18.2(e)), WLOG we can assume that f is surjective (onto). ASSUME $Z = A \cup B$, where A and B form a separation of Z. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint sets (since A and B are disjoint) whose union is X, which are open since f is continuous, and which are nonempty since f is onto. Therefore $f^{-1}(A)$ and $f^{-1}(B)$ are a separation of X. But this CONTRADICTS the hypothesis that X is connected. Hence there is no separation of $X = f(X)$ and so $f(X)$ is connected. □

Theorem 23.6

Theorem 23.6. A finite Cartesian product of connected spaces is connected.

Proof. We prove the result for two connected spaces X and Y and then the general result follows by induction.

Choose $(a, b) \in X \times Y$. Then $X \times \{b\}$ and $\{x\} \times Y$ are connected (for each $x \in X$), as will be shown in the homework (Exercise 23.A). For each $x \in X$, define $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$. Then T_x is connected by Theorem 23.3 (since (x, b) is in each constituent space). Next, consider $\cup_{x \in X} T_x = X \times Y$ (see Figure 23.2 on page 151 for motivation). This union is connected by Theorem 23.3 since the point (a, b) is common to each T_x. That is, $X \times Y$ is connected.

The proof for any finite product of connected spaces follows by induction along with the fact (established in Exercise 23.B) that $X_1 \times X_2 \times \cdots \times X_n$ is homeomorphic with $(x_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$. □