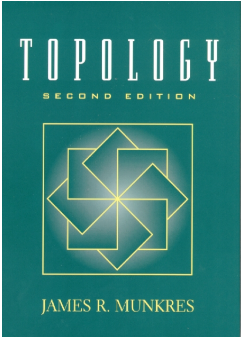
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 23. Connected Spaces—Proofs of Theorems



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Proof. Suppose that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is $\overline{A} \cap Y$ (where $\overline{A}$ denotes the closure of A in X) by Theorem 17.4. Since A is closed in Y, $Z = \overline{A} \cap Y$. Since $A \cap B = \varnothing$ then $\overline{A} \cap B = \varnothing$.

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Proof (continued). Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Then $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$ (again, since the closure of a set is the union of the set and its limit points by Theorem 17.6). Since $A \cup B = Y$, then $(\overline{A} \cap Y) \cup (\overline{B} \cap Y) = Y$. Since $A \cap (\overline{B} \cap Y) = \varnothing$ then we must have $\overline{A} \cap Y \subset A$ and similarly $\overline{B} \cap Y \subset B$.

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Lemma 23.2

Lemma 23.2. If sets C and D form a separation of X and if Y is a connected subspace of X, then Y lies entirely in either C or in D.

Proof. Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y. These two sets are disjoint and their union in Y.

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Theorem 23.3

Theorem 23.3. The union of a collection of connected subspaces of X that have a point in common is connected.

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Theorem 23.4. Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Proof. Let A be connected and let $A \subset B \subset \overline{A}$. ASSUME that $B = C \cup D$ where C and D are a separation of B. By Lemma 23.2, the set A must lie entirely in C or in D. WLOG, suppose $A \subset C$. Then $\overline{A} \subset \overline{C}$.

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Theorem 23.5

Theorem 23.5. The image of a connected space under a continuous map is connected.

Proof. Let $F : X \to Y$ be a continuous function where X is connected. Let $Z = f(X)$. Since the map obtained from f by restricting its range to the space Z is also continuous (by Theorem 18.2(e)), WLOG we can assume that f is surjective (onto).

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Choose $(a, b) \in X \times Y$. Then $X \times \{b\}$ and $\{x\} \times Y$ are connected (for each $x \in X$), as will be shown in the homework (Exercise 23.A). For each $x \in X$, define $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$. Then T_x is connected by Theorem 23.3 (since (x, b) is in each constituent space).

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The proof for any finite product of connected spaces follows by induction along with the fact (established in Exercise 23.B) that $X_1 \times X_2 \times \cdots \times X_n$ is homeomorphic with $(x_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$. □

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