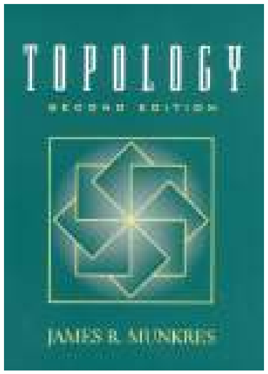
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 24. Connected Subspaces of the Real Line—Proofs of Theorems



Theorem 24.1

Theorem 24.1. If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Proof. Recall that a subspace Y of L is *convex* if for every pair of points $a, b \in Y$ with $a < b$, then entire interval $[a, b] = \{x \in L \mid a \leq x \leq b\}$ lies in Y.

Let Y be convex. ASSUME that Y has a separation and that Y is the union of disjoint nonempty sets A and B, each of which is open in Y. Choose $a \in A$ and $b \in B$. WLOG, say $a < b$. Since Y is convex then $[a, b] \subset Y$. Hence $[a, b]$ is the union of the disjoint sets $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$, each of which is open in $[a, b]$ in the subspace topology on $[a, b]$ (since A and B are open in Y) which is the same as the order topology (by Theorem 16.4). Since $a \in A_0$ and $b \in B_0$, then $A_0 \neq \varnothing \neq B_0$ and so A_0 and B_0 form a separation of $[a, b]$.

Theorem 24.1 (continued 1)

Proof (continued). Let $c = \sup A_0$. We now show in to cases that $c \notin A_0$ and $c \notin B_0$, which CONTRADICTS the fact that $[a, b] = A_0 \cup B_0$ (since $A_0 \subset [a, b]$ then b is an upper bound for A_0 and so $a \leq c \leq b$ and so $c \in [a, b] = A_0 \cup B_0$). From this contradiction, it follows that Y is connected.

Case 1. Suppose $c \in B_0$. Then $c \neq a$ (since $a \in A$ and $A \cap B = \varnothing$). So either $c = b$ or $a < c < b$. In either case, since B_0 is open in $[a, b]$ then there is some interval of the form $(d, c] \subset B_0$. If $c = b$ we have a contradiction since this implies that d is an upper bound of A_0, but $d < c$. If $c < b$ we note that $(c, d] \cap A_0 = \varnothing$ since c is an upper bound of A_0. Then (with d as above where $(d, c] \subset B_0$) we have that $(d, b] = (d, c] \cup (c, b]$ does not intersect A_0. Again, d is a smaller upper bound of A_0 than c, a CONTRADICTION. We conclude that $c \notin B_0$.

Theorem 24.1 (continued 2)

Theorem 24.1. If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Proof (continued).
Case 2. Suppose $c \in A_0$. Then $c \neq b$ since $b \in B$. So either $c = a$ of $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. By property (2) of the linear continuum L, there is $z \in L$ such that $c < z < e$. Then $z \in A_0$, CONTRADICTING the fact that c is an upper bound of A_0. We conclude that $c \notin A_0$.

We have shown that if Y is a convex subset of L then Y is connected. Notice that intervals and rays are convex sets and so are connected. □

Theorem 24.3. Intermediate Value Theorem

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Let $f : X \to Y$ be a continuum map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point $x \in X$ such that $f(c) = r$.

Proof. Suppose f, X, and Y are as hypothesized. The sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, +\infty)$ are disjoint (since $(-\infty, r)$ and $(r, +\infty)$ are disjoint) and nonempty since $f(a)$ is in one of these sets and $f(b)$ is in the other. Each is open in $f(X)$ under the subspace topology. ASSUME there is no point $c \in X$ such that $f(c) = r$. Then $f(X) = A \cup B$ and A and B form a separation of $f(X)$. But since X is connected and f is continuous then $f(X)$ is connected by Theorem 23.5, a CONTRADICTION. So the assumption that there is no such $c \in X$ is false and hence $f(c) = r$ for some $c \in X$. □

Theorem 24.A

Lemma 24.A. If space X is path connected then it is connected.

Proof. Let X be path connected. ASSUME X is not connected and that A and B form a separation of X. Let $f : [a, b] \to X$ be any path in X. Since f is continuous and $[a, b]$ is a connected set in $\mathbb{R}$, so by Theorem 23.5, $f([a, b])$ is connected in X. So by Lemma 23.2, $f([a, b])$ lies either entirely in A or entirely in B. But this cannot be the case if a is chosen from A and b is chosen from B, a CONTRADICTION. So the assumption that a separation of X exists is false and so space X is connected. □