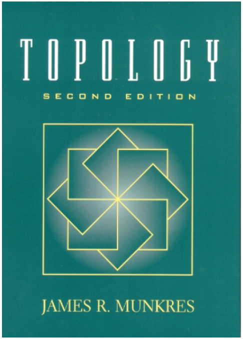
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 24. Connected Subspaces of the Real Line—Proofs of Theorems



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Let Y be convex. ASSUME that Y has a separation and that Y is the union of disjoint nonempty sets A and B, each of which is open in Y. Choose $a \in A$ and $b \in B$. WLOG, say $a < b$. Since Y is convex then $[a, b] \subset Y$.

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Proof (continued). Let $c = \sup A_0$. We now show in to cases that $c \notin A_0$ and $c \notin B_0$, which CONTRADICTS the fact that $[a, b] = A_0 \cup B_0$ (since $A_0 \subset [a, b]$ then b is an upper bound for A_0 and so $a \leq c \leq b$ and so $c \in [a, b] = A_0 \cup B_0$). From this contradiction, it follows that Y is connected.

Case 1. Suppose $c \in B_0$. Then $c \neq a$ (since $a \in A$ and $A \cap B = \varnothing$). So either $c = b$ or $a < c < b$. In either case, since B_0 is open in $[a, b]$ then there is some interval of the form $(d, c] \subset B_0$.

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Proof (continued).
Case 2. Suppose $c \in A_0$. Then $c \neq b$ since $b \in B$. So either $c = a$ of $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. By property (2) of the linear continuum L, there is $z \in L$ such that $c < z < e$. Then $z \in A_0$, CONTRADICTING the fact that c is an upper bound of A_0. We conclude that $c \notin A_0$.

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Theorem 24.3. Intermediate Value Theorem

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Let $f : X \to Y$ be a continuum map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point $x \in X$ such that $f(c) = r$.

Proof. Suppose f, X, and Y are as hypothesized. The sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, +\infty)$ are disjoint (since $(-\infty, r)$ and $(r, +\infty)$ are disjoint) and nonempty since $f(a)$ is in one of these sets and $f(b)$ is in the other.

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