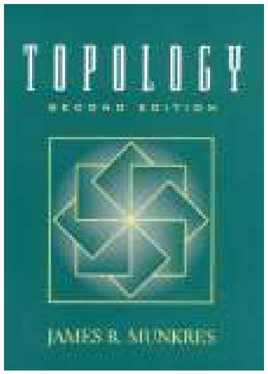
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 25. Components and Local Connectedness—Proofs of Theorems



Theorem 25.1

Theorem 25.1. The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Proof. Since the components are by definition equivalence classes, then the components are disjoint and union to give X (equivalence classes on a set partition the set; see page 23). ASSUME connected subspace A of X intersects two disjoint nonempty components C_1 and C_2, say at x_1 and x_2, respectively. Then $x_1 \sim x_2$ since $x_1, x_2 \in C_1$ and $x_1, x_2 \in C_2$. But since the components are disjoint then $C_1 = C_2$, a CONTRADICTION. So the assumption that a connected subspace can intersect two components is false.

Theorem 25.1 (continued)

Theorem 25.1. The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Proof (continued). To show that a component C is connected, let $x_0 \in C$. Then for each $x \in C$ we have $x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x. From the previous paragraph, a connected subspace cannot intersect two components and so $A_x \subset C$. Therefore, $C = \cup_{x \in C} A_x$. Since each A_x is connected and $x_0 \in A_x$ for all $x \in C$ then by Theorem 23.3, C is connected. □

Lemma 25.A

Lemma 25.A. Each connected component of a space X is closed. If X has only finitely many connected components, then each component of X is also open.

Proof. Let C be a connected component of X. By Theorem 23.4, $\overline{C}$ is also connected. Since the components are disjoint by Theorem 25.1, then $C = \overline{C}$ and so C is closed by Lemma 17.A.

If X has only finitely many components then the complement of a component C is a finite union of closed sets by the first part of this lemma, and so the complement of C is closed by Theorem 17.1. Hence C is open. □

Theorem 25.3

Theorem 25.3. A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof. Suppose X is locally connected. Let U be an open set in X, let C be a connected component of U, and let $x \in C$. Then by the definition of locally connected, there is a connected neighborhood V of x with $V \subset U$. Since V is connected, by Theorem 25.1, it must lie entirely in the component C, $V \subset C$. So C is open.

Conversely, suppose that the components of open sets in X are open. Let $x \in X$ and let U be an arbitrary neighborhood of x. Let C be the connected component of U which contains x. Now C is connected and, by hypothesis, open in X. So, by definition, X is locally connected. □

Theorem 25.5

Theorem 25.5. If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the component and the path components are the same.

Proof. Let C be a component of X. Let $x \in C$. Let P be the path component of X containing x. By Lemma 24.A, P is connected and so $P \subset C$ by Theorem 25.1, and the first claim holds.

Suppose X is locally path connected. ASSUME $P \neq C$. Let Q denote the union of all the path components of X that are different from P and which intersect C (since $P \neq C$ then $Q \neq \varnothing$). As above, by Lemma 24.A and Theorem 25.1, each of these path components must be in component C. Since the path components partition X by Theorem 25.2, then the path components in Q, along with path component P, partition C and $C = P \cup Q$. Since X is hypothesized to be locally path connected, then by Theorem 25.4 each path component of X is open in X.

Theorem 25.5 (continued)

Theorem 25.5. If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the component and the path components are the same.

Proof (continued). Therefore P (a path component of X) and Q (a union of path components) are both open in X and are disjoint by construction, so P and Q form a separation of C, a CONTRADICTION. So the assumption that $P \neq C$ is false and we have $P = C$. So arbitrary connected component C equals path component P. So every connected component of X is a path component and, since the connected components partition X by Theorem 25.1, conversely every path component is a connected component. □