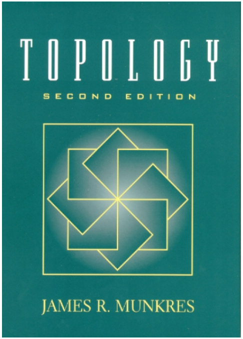
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 25. Components and Local Connectedness—Proofs of Theorems



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Proof (continued). To show that a component C is connected, let $x_0 \in C$. Then for each $x \in C$ we have $x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x. From the previous paragraph, a connected subspace cannot intersect two components and so $A_x \subset C$. Therefore, $C = \cup_{x \in C} A_x$. Since each A_x is connected and $x_0 \in A_x$ for all $x \in C$ then by Theorem 23.3, C is connected. □

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Lemma 25.A

Lemma 25.A. Each connected component of a space X is closed. If X has only finitely many connected components, then each component of X is also open.

Proof. Let C be a connected component of X. By Theorem 23.4, $\overline{C}$ is also connected. Since the components are disjoint by Theorem 25.1, then $C = \overline{C}$ and so C is closed by Lemma 17.A.

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If X has only finitely many components then the complement of a component C is a finite union of closed sets by the first part of this lemma, and so the complement of C is closed by Theorem 17.1. Hence C is open. □

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Theorem 25.3

Theorem 25.3. A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof. Suppose X is locally connected. Let U be an open set in X, let C be a connected component of U, and let $x \in C$.

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Proof. Suppose X is locally connected. Let U be an open set in X, let C be a connected component of U, and let $x \in C$. Then by the definition of locally connected, there is a connected neighborhood V of x with $V \subset U$. Since V is connected, by Theorem 25.1, it must lie entirely in the component C, $V \subset C$. So C is open.

Conversely, suppose that the components of open sets in X are open. Let $x \in X$ and let U be an arbitrary neighborhood of x. Let C be the connected component of U which contains x.

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Theorem 25.5. If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the component and the path components are the same.

Proof. Let C be a component of X. Let $x \in C$. Let P be the path component of X containing x. By Lemma 24.A, P is connected and so $P \subset C$ by Theorem 25.1, and the first claim holds.

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Suppose X is locally path connected. ASSUME $P \neq C$. Let Q denote the union of all the path components of X that are different from P and which intersect C (since $P \neq C$ then $Q \neq \varnothing$). As above, by Lemma 24.A and Theorem 25.1, each of these path components must be in component C.

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