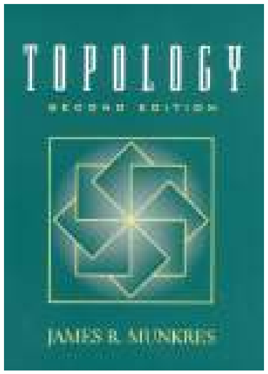
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 26. Compact Spaces—Proofs of Theorems



Lemma 26.1

Lemma 26.1. Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then the collection $\{A_\alpha \cap Y \mid \alpha \in J\}$ is a covering of Y by sets open in Y. Since Y is compact, there is a finite subcollection $\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$ covering Y. Then $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}$ that covers Y.

Lemma 26.1 (continued)

Lemma 26.1. Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof (continued). Conversely, suppose every covering of Y by sets open in X contain a finite subcollection covering Y. Let $\mathcal{A}' = \{A'_\alpha\}$ be an arbitrary covering of Y by sets open in Y. For each α, choose a set A_α open in X such that $A'_\alpha - A_\alpha \cap Y$ (this can be done since Y has the subspace topology and A'_α is open in Y. The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. Then by the hypothesis, some finite subcollection $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, A'_{\alpha_2}, \ldots, A'_{\alpha_n}\}$ is a subcollection of $\mathcal{A}'$ that covers Y. Therefore, Y is compact. □

Theorem 26.2

Theorem 26.2. Every closed subspace of a compact space is compact.

Proof. Let Y be a closed subspace of the compact set X. Let $\mathcal{A}$ be an arbitrary open cover of Y by sets open in X. Let $\mathcal{B} = \mathcal{A} \cup \{A \setminus Y\}$. Then $\mathcal{B}$ is a covering of X by open sets and since X is compact then some finite subcollection of $\mathcal{B}$ covers X. This finite subcollection with $X \setminus Y$ removed (if $X \setminus Y$ is in the subcollection) is then a finite subcollection of $\mathcal{A}$ which covers Y. So by Lemma 26.1, Y is compact. □

Lemma 26.4

Lemma 26.4. If Y is a compact subspace of the Hausdorff space X and $x_0 \notin Y$, then there exists disjoint open sets U and V of X containing x_0 and Y, respectively.

Proof. Since X is Hausdorff, then for each $y \in Y$ there are disjoint open U_y and V_y with $x_0 \in U_y$ and $y \in V_y$. Then $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X. Since Y is hypothesized to be compact, then by Lemma 26.1 there are finitely many elements of the covering which covers Y, say $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$. Define $V = V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}$ and $U = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}$. Then U and V are open, U and V are disjoint, $x_0 \in U$ and $Y \subset V$, as claimed. □

Theorem 26.3

Theorem 26.3. Every compact subspace of a Hausdorff space is closed.

Proof. Let Y be a compact subspace of Hausdorff space X. Let $x_0 \in X \setminus Y$. Then by Lemma 26.4 there is open $U \subset X \setminus Y$ with $x_0 \in U$. Therefore x_0 is an interior point of $X \setminus Y$ (by definition of interior of a set) and so $X \setminus Y = \text{int}(X \setminus Y)$ and by Lemma 17.A, $X \setminus Y$ is open and hence Y is closed. □

Theorem 26.5

Theorem 26.5. The image of a compact space under a continuous map is compact.

Proof. Let $f : X \to Y$ be continuous and X compact. Let $\mathcal{A}$ be an arbitrary covering of $f(X)$ by sets open in Y. Since f is continuous with domain X, the collection $\{f^{-1}(A) \mid A \in \mathcal{A}\}$ is a collection of open sets in X which covers X. Since X is compact, then there is finite subcollection $f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)$ which covers X. But then the finite subcollection $\{A_1, A_2, \ldots, A_n\}$ of $\mathcal{A}$ covers $f(X)$. So $f(X)$ is compact. □

Theorem 26.6

Theorem 26.6. Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. Since f is a bijection, then $f^{-1} : Y \to X$ is defined. Let $A \subset X$ be closed. Then by Theorem 26.2, A is compact. By Theorem 26.5, $f(A)$ is compact. Since Y is Hausdorff, by Theorem 26.3, $f(A)$ is closed. So f^{-1} is continuous by Theorem 18.1 (the (3)⇒(1) part). So f is a continuous bijection with a continuous inverse; that is, f is a homeomorphism. □

Lemma 26.8

Lemma 26.8. The Tube Lemma.
Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then N contains some "tube" $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X.

Proof. First, each element $\mathbf{x} \in \{x_0\} \times Y$ is an element of some basis element of the product topology. Since N is open and $\mathbf{x} \in N$, then $\mathbf{x}$ is in some basis element which is a subset of N by Lemma 13.1. So $\mathbf{x}$ is in a basis element of the product topology which is a subset of N (see part (2) of the definition of basis). Since $\{x_0\} \times Y$ is homeomorphic to Y and Y is compact, then $\{x_0\} \times Y$ is compact. So $\{x_0\} \times Y$ can be covered by finitely many of these sets, say $U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n$. WLOG, each of these sets intersects $\{x_0\} \times Y$ (otherwise, they can be eliminated from the covering). Define $W = U_1 \cap U_2 \cap \cdots \cap U_n$, so that W is open and $x_0 \in W$.

Lemma 26.8 (continued)

Lemma 26.8. The Tube Lemma.
Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then N contains some "tube" $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X.

Proof (continued). Now let $(x, y) \in W \times Y$. Consider $(x_0, y) \in \{x_0\} \times Y$. Then $(x_0, y) \in U_{i'} \times V_{i'}$ for some $i' = 1, 2, \ldots, n$, and so $y \in V_{i'}$ for some $i' = 1, 2, \ldots, n$. But $x \in W$ and so $x \in U_i$ for all $i = 1, 2, \ldots, n$. Therefore $(x, y) \in U_{i'} \times V_{i'}$. So $W \times Y \subset \cup_{i=1}^{n} U_i \times V_i \subset N$. So $W \times Y$ is the "tube" claimed to exist. □

Theorem 26.7

Theorem 26.7. The product of finitely many compact spaces is compact.

Proof. We prove the result for two spaces and then the general result follows by induction. Let X and Y be compact spaces and let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $\{x_0\} \times Y$ is compact since it is homeomorphic to Y. Hence $\{x_0\} \times Y$ can be covered by a finite number of elements of $\mathcal{A}$, say $A_1, A_2, \ldots, A_m$. Then $N = A_1 \cup A_2 \cup \cdots \cup A_m$ is an open set containing $\{x_0\} \times Y$. By The Tube Lemma (Lemma 26.8), there is a tube $W \times Y \subset N$ containing $\{x_0\} \times Y$ where W is open in X. So $W \times Y$ is covered by $A_1, A_2, \ldots, A_m$ of $\mathcal{A}$.

Theorem 26.7 (continued)

Theorem 26.7. The product of finitely many compact spaces is compact.

Proof (continued). Thus for each $x \in X$, there is W_x a neighborhood of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. Now the collection of all such W_x, $\{W_x \mid x \in X\}$, is an open covering of X; since X is compact, there is a finite subcollection $\{W_1, W_2, \ldots, W_k\}$ covering X. Then the union of tubes $W_1 \times Y, W_2 \times Y, \ldots, W_k \times Y$ is all of $X \times T$, $X \times T = \cup_{i=1}^{k} W_i \times Y$. Since each $W_i \times Y$ can be covered by finitely many elements of $\mathcal{A}$, then $X \times Y$ can be covered by finitely many elements of $\mathcal{A}$. Hence $X \times Y$ is compact and the result follows. □

Theorem 26.9

Theorem 26.9. Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\cap_{C \in \mathcal{C}} C$ for all elements of $\mathcal{C}$ is nonempty.

Proof. Given a collection $\mathcal{A}$ of subsets of X, let $\mathcal{C} = \{X \setminus A \mid A \in \mathcal{A}\}$. Then the following hold:

(1) $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.

(2) The collection $\mathcal{A}$ covers X if and only if the intersection $\cap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{V}$ is nonempty (since each $x \in X$ must be in some $A \in \mathcal{A}$ and so $x \notin X \setminus Z = C \in \mathcal{C}$).

(3) The finite subcollection $\{A_1, A_2, \ldots, A_n\} \subset \mathcal{A}$ covers X if and only if the intersection of the corresponding elements $C_i = X \setminus A_i$ of $\mathcal{C}$ is empty.

Theorem 26.9 (continued)

Proof (continued). The statement that X is compact is equivalent to:

> "Given any collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X then some finite subcollection of $\mathcal{A}$ covers X."

The (logically equivalent) contrapositive of this statement is:

> "Given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X."

This second statement can be rested using (1), (2), and (3) as:

> "Given any subcollection $\mathcal{C}$ of closed sets [by (1)], if every finite intersection of elements of $\mathcal{C}$ is nonempty [by (3)], then the intersection of all the elements of $\mathcal{C}$ is nonempty [by (2)]."

So the property of compactness of X is equivalent to the property involving collection of closed sets. □

Corollary 26.A

Corollary 26.A. Let X be a compact topological space and let $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ be a nested sequence of closed sets in X. If each C_n is nonempty, then $\cap_{n \in \mathbb{N}} C_n$ is nonempty.

Proof. For any finite collection of sets in $\mathcal{C}$, we have that the intersection equals C_N for some $N \in \mathbb{N}$, since the sets are nested, and $C_N \neq \varnothing$. So $\mathcal{C}$ has the finite intersection property. Since X is compact then, by Theorem 26.9, $\cap_{n \in \mathbb{N}} C_n$ is nonempty. □