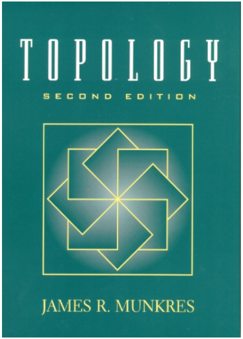
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 26. Compact Spaces—Proofs of Theorems



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Lemma 26.1. Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then the collection $\{A_\alpha \cap Y \mid \alpha \in J\}$ is a covering of Y by sets open in Y.

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Proof (continued). Conversely, suppose every covering of Y by sets open in X contain a finite subcollection covering Y. Let $\mathcal{A}' = \{A'_\alpha\}$ be an arbitrary covering of Y by sets open in Y. For each α, choose a set A_α open in X such that $A'_\alpha - A_\alpha \cap Y$ (this can be done since Y has the subspace topology and A'_α is open in Y. The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X.

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Theorem 26.2

Theorem 26.2. Every closed subspace of a compact space is compact.

Proof. Let Y be a closed subspace of the compact set X. Let $\mathcal{A}$ be an arbitrary open cover of Y by sets open in X. Let $\mathcal{B} = \mathcal{A} \cup \{A \setminus Y\}$.

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Proof. Let Y be a closed subspace of the compact set X. Let $\mathcal{A}$ be an arbitrary open cover of Y by sets open in X. Let $\mathcal{B} = \mathcal{A} \cup \{A \setminus Y\}$. Then $\mathcal{B}$ is a covering of X by open sets and since X is compact then some finite subcollection of $\mathcal{B}$ covers X. This finite subcollection with $X \setminus Y$ removed (if $X \setminus Y$ is in the subcollection) is then a finite subcollection of $\mathcal{A}$ which covers Y.

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Lemma 26.4

Lemma 26.4. If Y is a compact subspace of the Hausdorff space X and $x_0 \notin Y$, then there exists disjoint open sets U and V of X containing x_0 and Y, respectively.

Proof. Since X is Hausdorff, then for each $y \in Y$ there are disjoint open U_y and V_y with $x_0 \in U_y$ and $y \in V_y$. Then $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X.

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Theorem 26.5

Theorem 26.5. The image of a compact space under a continuous map is compact.

Proof. Let $f : X \to Y$ be continuous and X compact. Let $\mathcal{A}$ be an arbitrary covering of $f(X)$ by sets open in Y.

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Proof. Let $f : X \to Y$ be continuous and X compact. Let $\mathcal{A}$ be an arbitrary covering of $f(X)$ by sets open in Y. Since f is continuous with domain X, the collection $\{f^{-1}(A) \mid A \in \mathcal{A}\}$ is a collection of open sets in X which covers X. Since X is compact, then there is finite subcollection $f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)$ which covers X.

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Theorem 26.6

Theorem 26.6. Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. Since f is a bijection, then $f^{-1} : Y \to X$ is defined. Let $A \subset X$ be closed. Then by Theorem 26.2, A is compact. By Theorem 26.5, $f(A)$ is compact.

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Lemma 26.8

Lemma 26.8. The Tube Lemma.
Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then N contains some "tube" $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X.

Proof. First, each element $\mathbf{x} \in \{x_0\} \times Y$ is an element of some basis element of the product topology. Since N is open and $\mathbf{x} \in N$, then $\mathbf{x}$ is in some basis element which is a subset of N by Lemma 13.1.

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Proof (continued). Now let $(x, y) \in W \times Y$. Consider $(x_0, y) \in \{x_0\} \times Y$. Then $(x_0, y) \in U_{i'} \times V_{i'}$ for some $i' = 1, 2, \ldots, n$, and so $y \in V_{i'}$ for some $i' = 1, 2, \ldots, n$.

Lemma 26.8 (continued)

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Lemma 26.8 (continued)

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Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then N contains some "tube" $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X.

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Theorem 26.7

Theorem 26.7. The product of finitely many compact spaces is compact.

Proof. We prove the result for two spaces and then the general result follows by induction. Let X and Y be compact spaces and let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $\{x_0\} \times Y$ is compact since it is homeomorphic to Y.

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Theorem 26.7 (continued)

Theorem 26.7. The product of finitely many compact spaces is compact.

Proof (continued). Thus for each $x \in X$, there is W_x a neighborhood of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. Now the collection of all such W_x, $\{W_x \mid x \in X\}$, is an open covering of X; since X is compact, there is a finite subcollection $\{W_1, W_2, \ldots, W_k\}$ covering X. Then the union of tubes $W_1 \times Y, W_2 \times Y, \ldots, W_k \times Y$ is all of $X \times T$, $X \times T = \cup_{i=1}^{k} W_i \times Y$.

Theorem 26.7 (continued)

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Proof (continued). Thus for each $x \in X$, there is W_x a neighborhood of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. Now the collection of all such W_x, $\{W_x \mid x \in X\}$, is an open covering of X; since X is compact, there is a finite subcollection $\{W_1, W_2, \ldots, W_k\}$ covering X. Then the union of tubes $W_1 \times Y, W_2 \times Y, \ldots, W_k \times Y$ is all of $X \times T$, $X \times T = \cup_{i=1}^k W_i \times Y$. Since each $W_i \times Y$ can be covered by finitely many elements of $\mathcal{A}$, then $X \times Y$ can be covered by finitely many elements of $\mathcal{A}$. Hence $X \times Y$ is compact and the result follows. □

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Theorem 26.9

Theorem 26.9. Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\cap_{C \in \mathcal{C}} C$ for all elements of $\mathcal{C}$ is nonempty.

Proof. Given a collection $\mathcal{A}$ of subsets of X, let $\mathcal{C} = \{X \setminus A \mid A \in \mathcal{A}\}$.

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Proof. Given a collection $\mathcal{A}$ of subsets of X, let $\mathcal{C} = \{X \setminus A \mid A \in \mathcal{A}\}$. Then the following hold:

(1) $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.

(2) The collection $\mathcal{A}$ covers X if and only if the intersection $\cap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{V}$ is nonempty (since each $x \in X$ must be in some $A \in \mathcal{A}$ and so $x \notin X \setminus Z = C \in \mathcal{C}$).

(3) The finite subcollection $\{A_1, A_2, \ldots, A_n\} \subset \mathcal{A}$ covers X if and only if the intersection of the corresponding elements $C_i = X \setminus A_i$ of $\mathcal{C}$ is empty.

Theorem 26.9

Theorem 26.9. Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\cap_{C \in \mathcal{C}} C$ for all elements of $\mathcal{C}$ is nonempty.

Proof. Given a collection $\mathcal{A}$ of subsets of X, let $\mathcal{C} = \{X \setminus A \mid A \in \mathcal{A}\}$. Then the following hold:

(1) $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.

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Theorem 26.9 (continued)

Proof (continued). The statement that X is compact is equivalent to:

> "Given any collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X then some finite subcollection of $\mathcal{A}$ covers X."

The (logically equivalent) contrapositive of this statement is:

> "Given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X."

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This second statement can be rested using (1), (2), and (3) as:

> "Given any subcollection $\mathcal{C}$ of closed sets [by (1)], if every finite intersection of elements of $\mathcal{C}$ is nonempty [by (3)], then the intersection of all the elements of $\mathcal{C}$ is nonempty [by (2)]."

Theorem 26.9 (continued)

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So the property of compactness of X is equivalent to the property involving collection of closed sets. □

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Corollary 26.A

Corollary 26.A. Let X be a compact topological space and let $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ be a nested sequence of closed sets in X. If each C_n is nonempty, then $\cap_{n \in \mathbb{N}} C_n$ is nonempty.

Proof. For any finite collection of sets in $\mathcal{C}$, we have that the intersection equals C_N for some $N \in \mathbb{N}$, since the sets are nested, and $C_N \neq \varnothing$. So $\mathcal{C}$ has the finite intersection property. Since X is compact then, by Theorem 26.9, $\cap_{n \in \mathbb{N}} C_n$ is nonempty. □

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