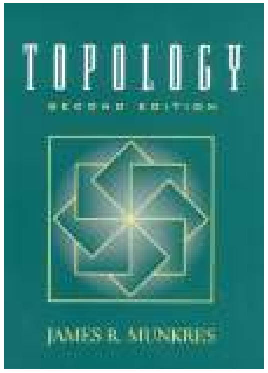
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 27. Compact Subspaces of the Real Line—Proofs of Theorems



Theorem 27.1

Theorem 27.1. Let X be a simply ordered set having the least upper bound property. In the order topology, each closed and bounded interval X is compact.

Proof. We follow Munkres' 4-step proof.
Step 1. Let $a < b$ and let $\mathcal{A}$ be a covering of a, b be sets open in $[a, b]$ in the subspace topology (which is the same as the order topology, by Theorem 16.4). Let $x \in [a, b]$, $x \neq b$. If x has an immediate successor in X, let y be this immediate successor. Then $[x, y] = \{x, y\}$ and $[x, y]$ can be covered by at most two elements of $\mathcal{A}$. If x has no immediate successor in X, choose an element $A \in \mathcal{A}$ containing x. Because $x \neq b$ and A is open, A contains an interval of the form $[x, c)$ for some $x \in [a, b]$ (since this is an element of the basis for the order topology; see part (2) of the definition of "order topology"). Choose $y \in (x, c)$. Then the interval $[a, y]$ is covered by the single element A of $\mathcal{A}$. In either case, for each $x \in [a, b)$ there is $y > x$ where $y \in [a, b]$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

Theorem 27.1 (continued 1)

Proof (continued).
Step 2. Let $C = \{y \in [a, b] \mid y > a$ and $[a, y]$ can be covered by finitely many elements of $\mathcal{A}\}$. Since $a \in C$, $C \neq \varnothing$. Let c be the least upper bound of set C (this is where the least upper bound property is used). Then, by Step 1, $a < c \leq b$.

Step 3. Since $\mathcal{A}$ is a covering of $[a, b]$, then some $A \in \mathcal{A}$ contains c. Since A is open, it contains an interval of the form $(d, c]$ for some $d \in [a, b]$ (see part (3) of the definition of "order topology"). ASSUME $c \notin C$. Then there must be $x \in C$ with $z \in (d, c)$, otherwise $d < c$ would be an upper bound on C. Since $z \in C$, the interval $[a, z]$ can be covered by finitely many (say n) elements of $\mathcal{A}$ (by the definition of C). Now $[z, c] \subset (d, c] \subset A \in \mathcal{A}$, hence $[a, c] = [a, z] \cup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$. But then $c \in C$, a CONTRADICTION. So the assumption that $c \notin C$ is false, and in fact $c \in C$.

Theorem 27.1 (continued 2)

Theorem 27.1. Let X be a simply ordered set having the least upper bound property. In the order topology, each closed and bounded interval X is compact.

Proof (continued).
Step 4. ASSUME $c < b$ where $c = \text{lub}(C)$, as defined in Step 2. By Step 1 with $x = c$, there is $y \in [a, b]$ with $y > c$ such that $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$. From Step 3, $c \in C$ and so $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. So $y \in C$. But $y > c$, CONTRADICTING the fact that c is an upper bound of C. So the assumption that $c < b$ is false and so $c = b$ (notice $c \leq b$ by Step 2). By Step 3, $b = c \in C$ and so the interval $[a, b]$ can be covered by finitely many elements of $\mathcal{A}$. Since $\mathcal{A}$ is an arbitrary open covering of $[a, b]$, then $[a, b]$ is compact. □

Corollary 27.A

Corollary 27.A. Every closed and bounded set in $\mathbb{R}$ (where $\mathbb{R}$ has the standard topology) is compact.

Proof. Let set $C \subset \mathbb{R}$ be closed and bounded. Then $c \in [a, b]$ for some $a, b \in \mathbb{R}$. Since $[a, b] \subset \mathbb{R}$ is compact by Corollary 27.2, then by Theorem 26.2 C is compact (since C is closed on $[a, b]$). □

Theorem 27.3

Theorem 27.3. The Heine-Borel Theorem.
A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the Euclidean metric d or the square metric ρ.

Proof. Recall from Theorem 20.3 that the topologies on $\mathbb{R}^n$ induced by d and ρ are the same as the product topology (and the box topology) on $\mathbb{R}^n$. In the proof of Theorem 20.3, the inequality $\rho(\mathbf{x}, \mathbf{y}) \le d(\mathbf{x}, \mathbf{y}) \le \sqrt{n}\rho(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is established, so that A is bounded under d if and only if A is bounded under ρ.

Suppose that A is compact. Since $\mathbb{R}^n$ is Hausdorff, by Theorem 26.3, A is closed. Consider the collection of open sets $\{B_\rho(\mathbf{0}, m) \mid m \in \mathbb{N}\}$, whose union is all of $\mathbb{R}^n$. Since A is compact, some finite subcollection covers A and so $A \subset B_\rho(\mathbf{0}, M)$ for some $M \in \mathbb{N}$. So (by the Triangle Inequality) for any $\mathbf{x}, \mathbf{y} \in A$ we have $\rho(\mathbf{x}, \mathbf{y}) \le 2M$ and hence A is bounded under ρ (see the definition "bounded" in Section 20). That is, A is closed and bounded.

Theorem 27.3

Theorem 27.3. The Heine-Borel Theorem.
A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the Euclidean metric d or the square metric ρ.

Proof. Conversely, suppose that A is closed and bounded under ρ, say $\rho(\mathbf{x}, \mathbf{y}) \le N$ for all $\mathbf{x}, \mathbf{y} \in A$. Let $\mathbf{x}_0 \in A$ be given and let $\rho(\mathbf{x}_0, \mathbf{0}) = b$. By the Triangle Inequality, every $\rho(\mathbf{x}, \mathbf{0}) \le \mathbf{x}, \mathbf{x}_0) + \rho(\mathbf{x}_0, \mathbf{0}) = N + b$ for every $\mathbf{x} \in A$. Set $P = N + b$ and then A is a subset of the cube $[-P, P]^n$, which is compact by Theorem 26.7 since each $[-P, P] \subset \mathbb{R}$ is compact (by Corollary 27.2). Since $A \subset [-P, P]^n$ is closed, A is compact by Theorem 26.2. □

Theorem 27.4

Theorem 27.4. Extreme Value Theorem.
Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exists points $c, d \in X$ such that $f(c) \le f(x) \le f(d)$ for all $x \in X$.

Proof. Since f is continuous and X is compact, then by Theorem 26.5, $A = f(X)$ is compact.

ASSUME that A has no largest element. Then the collection $\{(-\infty, a) \mid a \in A\}$ is an open covering of A and so has some finite subcover $\{(-\infty, a_1), (-\infty, a_2), \ldots, (-\infty, a_n)\}$. Let $a_i = \max\{a_1, a_2, \ldots, a_n\}$. Then $a_i \in A$ but a_i is not covered by the subcollection, a CONTRADICTION. So A does in fact have a largest element M. Similarly, A has a least element m. Then there are $c, d \in X$ such that $f(c) = m \le f(x) \le M = f(d)$ for all $x \in X$. □

Lemma 27.A

Lemma 27.A. Let (X, d) be a metric space and A a fixed subset of X. Then $D : X \to \mathbb{R}$ defined as $D(x) = d(x, A)$ is continuous.

Proof. Let $x \in X$ and $\varepsilon > 0$. Let $\delta = \varepsilon$. If $y \in X$ and $d(x, y) < \delta$ then for $a \in A$ we have $d(x, A) \le d(x, a) \le d(x, y) + d(y, a)$ (by the previous definition and the Triangle Inequality). Hence $d(x, A) - d(x, y) \le d(y, a) \le d(y, A)$ or $d(x, A) - d(y, A) \le d(x, y) < \varepsilon$. Similarly (interchanging x and y), $d(y, A) - d(x, A) \le d(x, y) < \varepsilon$ and so $|d(x, A) - d(y, A)| = |D(x) - D(y)| < \varepsilon$. Hence D is continuous on X. □

Theorem 27.5

Lemma 27.5. The Lebesgue Number Lemma.
Let $\mathcal{A}$ be an open covering of metric space (X, d). If X is compact, there is $\delta > 0$ such that for each subset B of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing B. The number $\delta > 0$ is a Lebesgue number for covering $\mathcal{A}$.

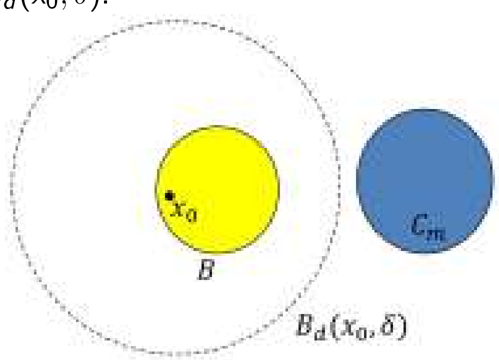
Proof. Let $\mathcal{A}$ be an open covering of X. If X itself is an element of $\mathcal{A}$ then any $\delta > 0$ is a Lebesgue number for $\mathcal{A}$, so WLOG, $X \notin \mathcal{A}$. Since X is compact, there is a finite subcollection $\{A_1, A_2, \ldots, A_n\} \subset \mathcal{A}$ that covers X. Set $C_i = X \setminus A_i$ for $i = 1, 2, \ldots, n$ and define $f : X \to \mathbb{R}$ as

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i)$$

(the average of the $d(x, C_i)$). For arbitrary $x \in X$, we have $x \in A_i$ for some i. Since A_i is open and (X, d) is a metric space, for some $\varepsilon > 0$, $B_d(x, \varepsilon) \subset A_i$ and so $d(x, C_i) \ge \varepsilon$ and so $f(x) \ge \varepsilon/n > 0$. That is, f is positive on X.

Theorem 27.5 (continued 1)

Proof (continued). Since f is continuous by Lemma 27.A (and the fact that a sum of real valued continuous functions is continuous) and X is compact by hypothesis, by Theorem 27.4 (the Extreme Value Theorem) f attains a minimum value $\delta > 0$ on X (this is where the fact that f is positive is used). Let B be a subset of X of diameter less than δ and let $x_0 \in B$. Then $B \subset B_d(x_0, \delta)$.



Theorem 27.5 (continued 2)

Lemma 27.5. The Lebesgue Number Lemma.
Let $\mathcal{A}$ be an open covering of metric space (X, d). If X is compact, there is $\delta > 0$ such that for each subset B of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing B. The number $\delta > 0$ is a Lebesgue number for covering $\mathcal{A}$.

Proof (continued). Now

$$\begin{aligned} \delta &\le f(x_0) \text{ since } \delta \text{ is the minimum of } f \text{ on } X \\ &= \frac{a}{n}\sum_{i=1}^{n} d(x_0, C_i) \\ &\le d(x_0, C_M) \end{aligned}$$

where $d(x_0, C_m)$ is the largest of the numbers $d(x_0, C_i)$. Then the δ-neighborhood $B_d(x_0, \delta)$ of x_0 is contained in the element $A_m = X \setminus C_m \in \mathcal{A}$. Since $B \subset X$ of diameter less than δ is arbitrary, then δ is a Lebesgue number for $\mathcal{A}$. □

Theorem 27.6

Theorem 27.6. Uniform Continuity Theorem.
Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space $Y, d_Y)$. Then f is uniformly continuous on X.

Proof. Let $\varepsilon > 0$. Consider the open covering of Y of $\{B_{d_Y}(y, \varepsilon/2) \mid y \in Y\}$. Since f is continuous, each $f^{-1}(B_{d_Y}(y, \varepsilon/2))$ is open. Let $\mathcal{A}$ be the open covering of X of $\mathcal{A} = \{f^{-1}(B_{d_Y}(y, \varepsilon/2)) \mid y \in Y\}$. Since X is a compact metric space, then $\mathcal{A}$ has a Lebesgue number δ by Lemma 27.5 (The Lebesgue Number Lemma). Then if $x_a, x_1 \in X$ with $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has diameter less than δ so that $\{x_1, x_2\}$ is a subset of some element of $\mathcal{A}$ and so $\{f(x_1), f(x_2)\}$ lies in some $B_{d_Y}(y, \varepsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \varepsilon$. So f is uniformly continuous. □

Theorem 27.7

Theorem 27.7. Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.

Proof. We follow the two-step proof of Munkres.
Step 1. Let U be a nonempty open set of X and let $x \in X$. Let $y \in U$ where $y \neq x$ (this is possible since X has no isolated points and so if $x \in U$ then $|U| \neq 1$; if $x \notin U$ this is possible since $U \neq \varnothing$). Since X is Hausdorff, there are disjoint open W_1 and W_2 with $x \in W_1$ and $y \in W_2$. Let $V = W_2 \cap U$. Then V is open, $y \in V$, $v \neq \varnothing$, and $V \subset U$. Since W_1 is a neighborhood of x which does not intersect V, then x is not a limit point of V (by the definition of "limit point"). By Theorem 17.6, $\overline{A} = A \cup A'$ (where A' is the set of limit points of A) so $x \notin \overline{V}$. Therefore, for any open set U and any $x \in X$ there is a nonempty set $V \subset U$ such that $x \notin \overline{V}$.

Theorem 27.7 (continued)

Theorem 27.7. Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.

Proof (continued).
Step 2. Suppose $f : \mathbb{N} \to X$. Denote $x_n = f(x_n)$. By Step 1 applied to nonempty open set $U = X$, there is a nonempty open set $V_1 \subset X$ such that $x_1 \notin \overline{V}_1$. Inductively, define V_{n+1} given V_n by applying Step 1 to nonempty open set $V_n = U$ to produce nonempty open set $V_{n+1} \subset U = V_n$ such that $x_{n+1} \notin \overline{V}_{n+1}$. Then we have the sequence of nonempty, nested, closed sets $\overline{V}_1 \supset \overline{V}_2 \supset \overline{V}_3 \supset \cdots$. Since X is compact, by Theorem 26.9 there is some $x \in \cap_{m=1}^{\infty} \overline{V}_m$. Notice that for each x_n, we have $x_n \notin \overline{V}_n$ and so $x_n \notin \cap_{m=1}^{\infty} \overline{V}_m$ for all $n \in \mathbb{N}$. So x is different from all x_n and so there is no $n \in \mathbb{N}$ such that $f(n) = x$. Therefore, arbitrary function $f : \mathbb{N} \to X$ is not surjective (onto). Hence, $|\mathbb{N}| < |X|$ and X is uncountable. □