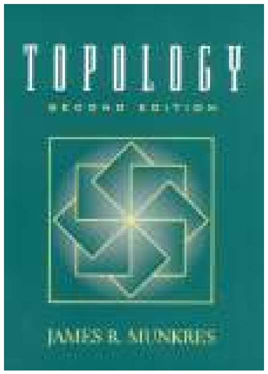
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 28. Limit Point Compactness—Proofs of Theorems



Theorem 28.1

Theorem 28.1. Compactness implies limit point compactness, but not conversely.

Proof. Let X be compact and let $A \subset X$. We prove the (logically equivalent) contrapositive of the claim: If A has no limit point, then A must be finite.

Suppose $A \subset X$ has no limit point. Then A contains all of its limit points and so A is closed by Corollary 17.7. Furthermore, for each $a \in A$, since a is not a limit point of A, there is a neighborhood U_a of a such that U_a intersects A in the point a only. The space X is covered by the open set $X \setminus A$ and each open U_a. Since X is compact, it can be covered by finitely many of these sets. Since $X \setminus A$ does not intersect A, each U_a contains only one point of A and so set A is finite. So if X is compact then it is limit point compact. The fact that the converse does not necessarily hold is given in the following example. □

Lemma 28.A

Lemma 28.A. Let X be metrizable. If X is also sequentially compact then the conclusion of the Lebesgue Number Lemma (Lemma 27.5) holds for X.

Proof. Let $\mathcal{A}$ be an open covering of X. ASSUME there is no Lebesgue number for open covering $\mathcal{A}$. That is, assume there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it.

So for each $n \in \mathbb{N}$, there is a set of diameter less than $1/n$ that is not contained in any element of $\mathcal{A}$. Let C_n be such a set. Choose $x_n \in C_n$ for each $n \in \mathbb{N}$. Since X is hypothesized to be sequentially compact, then some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges, say to point a. Now $a \in A$ for some $A \in \mathcal{A}$. Since A is open and X is metrizable, there is $\varepsilon > 0$ such that $B(a, \varepsilon) \subset A$. With i large enough so that $1/n_i < \varepsilon/2$, then the set C_{n_i} has diameter less than $1/n_i < \varepsilon/2$ and so $C_{n_i} \subset B(x_{n_i}, \varepsilon/2)$.

Lemma 28.A (continued)

Lemma 28.A. Let X be metrizable. If X is also sequentially compact then the conclusion of the Lebesgue Number Lemma (Lemma 27.5) holds for X.

Proof (continued). With i large enough so that $d(x_{n_i}, a) < \varepsilon/2$ (which can be done sine $\{x_{n_i}\} \to a$), then $C_n \subset B(a, \varepsilon \subset A$. But this CONTRADICTS the assumption that $\mathcal{A}$ has no Lebesgue number (and the implication of that assumption that such C_b exists which is *not* a subset of some element of $\mathcal{A}$). Therefore there is a Lebesgue number for $\mathcal{A}$. Since $\mathcal{A}$ is an arbitrary open covering of X, then X satisfies the conclusion of the Lebesgue Number Lemma (Lemma 27.5). □

Lemma 28.B

Lemma 28.B. Let X be metrizable. If X is also sequentially compact, then for all $\varepsilon > 0$ there exists a finite covering of X by open ε-balls.

Proof. ASSUME that, to the contrary of the claim, there is $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls. Construct sequence $\{x_n\{$ as follows: First, let $x_1 \in X$ be any point in X. By assumption, $B(x_1, \varepsilon)$ is not all of X, so there is $x_2 \in X \setminus B(a_1, \varepsilon)$. Inductively, let $x_{n+1} \in X \setminus (B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon))$; such x_{n+1} exists since the n ε-balls are assumed to not cover X. By construction, $d(x_{n+1}, x_i) \geq \varepsilon$ for all $i = 1, 2, \ldots, n$. So any $\varepsilon/2$-ball in X can contain either one or no elements of the sequence $\{x_n\}$. Hence $\{x_n\}$ can have no convergent subsequence. But this CONTRADICTS the sequential compactness of X. So the claim holds. □

Theorem 28.2

Theorem 28.2. Let X be a metrizable space. Then the following are equivalent:

(1) X is compact.
(2) X is limit point compact.
(3) X is sequentially compact.

Proof. (1)⇒(2): This follows from Theorem 28.1.

(2)⇒(3): Suppose X is limit point compact. Let $\{x_n\}$ be a sequence of points of X. Consider the set $A = \{x_n \mid n \in \mathbb{N}\}$. If set A is finite, then there is at least one point x such that $x = x_n$ for infinitely many values $n \in \mathbb{N}$. In this case, $\{x_n\}$ has a subsequence that is constant and hence convergent. On the other hand, if A is infinite, then A has a limit point x since X is hypothesized to be limit point compact.

Theorem 28.2 (continued)

Proof(continued). We define a subsequence of $\{x_n\}$ converging to x as follows: Let n_1 be such that $x_{n_1} \in B(x, 1)$. Inductively define n_{i+1} in terms of n_i by letting n_{i+1} be such that $B(x, 1/i)$ contains $x_{n_{i+1}}$ and $n_{i+1} > n_i$ (such n_{i+1} exists since $B(x, 1/i)$ contains infinitely many points of A). Then the subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ converges to x. Since $\{x_n\}$ is an arbitrary sequence in X, then X is sequentially compact.

(3)⇒(1): Let $\mathcal{A}$ be an open covering of sequentially compact metrizable X. Then by Lemma 28.A, covering $\mathcal{A}$ has a Lebesgue number δ. Let $\varepsilon = \delta/3$. By Lemma 28.B, there is a finite covering of X with open ε-balls. Each of these balls has diameter at most $2\delta/3 < \delta$ and so each ball lies in an element of $\mathcal{A}$. Choose one element of $\mathcal{A}$ for each of these finite number of ε-balls and, since the ε-balls cover X, then finite subcollection of $\mathcal{A}$ covers X. Since open covering $\mathcal{A}$ is arbitrary, then X is compact. □