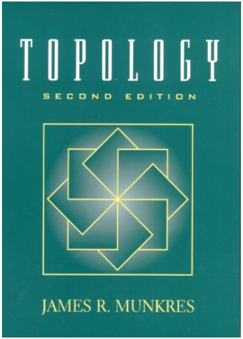
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 28. Limit Point Compactness—Proofs of Theorems



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Suppose $A \subset X$ has no limit point. Then A contains all of its limit points and so A is closed by Corollary 17.7. Furthermore, for each $a \in A$, since a is not a limit point of A, there is a neighborhood U_a of a such that U_a intersects A in the point a only.

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Lemma 28.A

Lemma 28.A. Let X be metrizable. If X is also sequentially compact then the conclusion of the Lebesgue Number Lemma (Lemma 27.5) holds for X.

Proof. Let $\mathcal{A}$ be an open covering of X. ASSUME there is no Lebesgue number for open covering $\mathcal{A}$. That is, assume there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it.

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So for each $n \in \mathbb{N}$, there is a set of diameter less than $1/n$ that is not contained in any element of $\mathcal{A}$. Let C_n be such a set. Choose $x_n \in C_n$ for each $n \in \mathbb{N}$.

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Proof (continued). With i large enough so that $d(x_{n_i}, a) < \varepsilon/2$ (which can be done sine $\{x_{n_i}\} \to a$), then $C_n \subset B(a, \varepsilon \subset A$. But this CONTRADICTS the assumption that $\mathcal{A}$ has no Lebesgue number (and the implication of that assumption that such C_b exists which is *not* a subset of some element of $\mathcal{A}$). Therefore there is a Lebesgue number for $\mathcal{A}$. Since $\mathcal{A}$ is an arbitrary open covering of X, then X satisfies the conclusion of the Lebesgue Number Lemma (Lemma 27.5). □

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Lemma 28.B

Lemma 28.B. Let X be metrizable. If X is also sequentially compact, then for all $\varepsilon > 0$ there exists a finite covering of X by open ε-balls.

Proof. ASSUME that, to the contrary of the claim, there is $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls.

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Theorem 28.2. Let X be a metrizable space. Then the following are equivalent:

(1) X is compact.

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