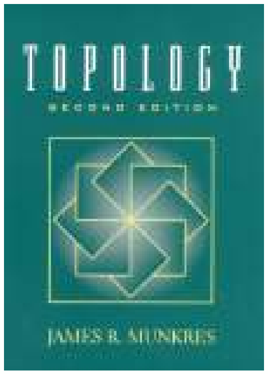
Introduction to Topology

Chapter 3. Connectedness and Compactness
Section 29. Local Compactness—Proofs of Theorems



Theorem 29.1

Theorem 29.1. Let X be a topological space. Then X is a locally compact Hausdorff space if and only if there is a topological space Y satisfying the following conditions:

(1) X is a subspace of Y.

(2) The set $Y \setminus X$ consists of a single point.

(3) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof. We follow Munkres' three-step proof (which oddly enough does not correspond to the numbered conditions of Y).

Step 1. We first verify the homeomorphism claim. Let Y and Y' be spaces satisfying the three conditions. Define $Y \to Y'$ by letting h map the "single point" $p \in Y \setminus X$ to the "single point" $q \in Y' \setminus X$, and letting h equal the identity on X. Then h is a bijection (one to one and onto).

Theorem 29.1 (continued 1)

Proof (continued). Let U be an open set in Y. First, suppose $p \notin U$. Then $h(U) = U$ is open in X (under the subspace topology). Now X is open in Y' since $Y \setminus X$ is closed ($Y' \setminus X$ is a singleton, which forms a closed set, by Theorem 17.8, because Y is compact by (3)) and so U is open in Y'. Second, suppose $p \in U$. Since $C = Y \setminus U$ is closed in Y, then C is a compact subspace of Y, by Theorem 26.2, since Y is compact by (3). Since $C \subset X$, C is also compact in X. Since $X \subset Y'$, the space C is also a compact subspace of Y' (every open covering of C with sets open in Y' yields an open covering of C with sets open in X under the subspace topology—and hence finite subcovers). Since Y' is Hausdorff by (3), Theorem 26.3 implies that C is closed in Y', and so $Y' \setminus C$ is open. But $h(U) = U \cup \{q\} = (Y \setminus C) \cup \{q\} = Y' \setminus C$ and so $h(U)$ is open. In both cases, for any open U we have that $h(U)$ is open and so h^{-1} is continuous. Interchanging Y and Y' shows that h is continuous and therefore h is a homeomorphism.

Theorem 29.1 (continued 2)

Proof (continued).
Step 2. Suppose X is locally compact and Hausdorff. We construct set Y by adding a single element to X, say $Y = X \cup \{\infty\}$. This give condition (2). Define the collection of subsets of Y, $\mathcal{T} = T_1 \cup T_2$ where $T_1 = \{U \subset X \mid U \text{ is open in } X\}$ and $T_2 = \{Y \setminus C \mid C \subset X \text{ is compact in } X\}$. We now show that $\mathcal{T}$ is a topology on Y. Since $\varnothing$ is open and compact in X, then $\varnothing, Y \in \mathcal{T}$. For closure of $\mathcal{T}$ under intersections we consider three cases:

$$U_1 \cap U_2 \in T_1$$

$$(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) \in T_2$$

$$U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1) \in T_1.$$

Theorem 29.1 (continued 3)

Proof (continued). Similarly, we have closure under unions:

$$\cup U_\alpha = U \in \mathcal{T}_1$$

$$\cup(Y \setminus C_\beta) = Y \setminus (\cap C_\beta) = V \setminus C \in \mathcal{T}_2$$

$$(\cup U_\alpha) \cup (\cup Y \setminus C_\beta) = U \cup (Y \setminus C) = T \setminus (C \setminus U) \in \mathcal{T}_2.$$

Now we show that X is a subspace of Y (confirming condition (1)). Given any open set U of Y, we need to show that $X \cap U$ is open in X. If $U \in \mathcal{T}_1$ then $U \cap X = U$; if $U = Y \setminus C \in \mathcal{T}_2$ then $(Y \setminus C) \cap X = X \setminus C \in \mathcal{T}_2$. Conversely, any open set in X is in $\mathcal{T}_1$ and therefore is open in Y. So the topology on X is the same as the subspace topology on X as a subspace of Y. That is, X is a subspace of Y and condition (1) holds.

Theorem 29.1 (continued 4)

Proof (continued). Now we show that Y is compact. Let $\mathcal{A}$ be an open covering of Y. Since ∞ must be in some element of $\mathcal{A}$, then there is compact $C \subset X$ such that $Y \setminus C \in \mathcal{T}_2$ is in $\mathcal{A}$. Since C is compact and $\mathcal{A}$ is a covering of C then there is a finite subcover $\mathcal{A}'$ of $\mathcal{A}$ which covers C. Then $\mathcal{A}' \cup \{Y \setminus C\}$ is a finite cover of C. Then $\mathcal{A}' \cup \{Y \setminus C\}$ is a finite cover of Y. Hence Y is compact.

Next, we show that Y is Hausdorff. Let $x, y \in Y$ with $x \neq y$. If x and y are both in X, then there are disjoint open sets U and V in X containing x and y, respectively, since X is Hausdorff. If $x \in X$ and $y = \infty$ then, since X is hypothesized to be locally compact, there is compact C in X containing neighborhood U of x. Then U and $Y \setminus C$ are disjoint open sets containing x and $y = \infty$, respectively. So Y is Hausdorff. Hence, condition (3) holds.

Theorem 29.1 (continued 5)

Proof (continued).
Step 3. We now show the converse. Suppose Y satisfies conditions (1), (2), and (3). Then X is Hausdorff because it is a subspace of Y (and it has the subspace topology). Let $x \in X$ be given. Since Y is Hausdorff, there are disjoint open sets U and V in Y containing ∞ and the single point of $Y \setminus X = \{\infty\}$, respectively. The set $C = Y \setminus V$ is closed in Y and so is compact since Y is compact (by Theorem 26.2). Since $\infty \in V$ then $\infty \notin C = Y \setminus V$ and so $C \subset X$ is also compact in X (since X has the subspace topology by (1). Also, C contains neighborhood U of x, and so X is locally compact. □

Theorem 29.2

Theorem 29.2. Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof. If X satisfies this condition, then certainly there is a compact subspace of X (namely $\overline{V}$) containing a neighborhood V of x; that is, the condition implies locally compact.

Conversely, suppose X is locally compact and let $x \in X$ with U a neighborhood of x. Since S is locally compact, by Theorem 29.1 there is a space Y, the one-point compactification of X. Let $C = Y \setminus U$. Since U is open in X then U is open in Y (in the proof of Theorem 29.1, all sets open in X are open in Y) and so C is closed in Y. Since Y is compact, then by Theorem 26.2, C is compact in Y. By Lemma 26.4 there are disjoint open sets V and W containing x and C, respectively.

Theorem 29.2 (continued)

Theorem 29.2. Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof (continued). Then $\overline{V}$ is compact (again, by Theorem 26.2) and $\overline{V}$ is disjoint from C since $\overline{V} = V \cup V'$ where V' is the set of limit point of set V, and since $x \in V$, $C \subset W$, and $V \cap W = \varnothing$, then no points of C are points of closure of V. So $\overline{V} \subset T \setminus C = U$ is the desired set. □

Corollary 29.3

Corollary 29.3. Let X by locally compact and Hausdorff. Let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Proof. Suppose A is closed in X. Given $x \in A$, let C be a compact subspace of X containing neighborhood U of $x \in X$ (which can be done since X is locally compact). Then $C \cap A$ is closed in C and thus (by Theorem 26.2) compact and it contains the neighborhood $U \cap A$ of $x \in A$. That is, A is locally compact.

Suppose A is open in X. Let $x \in A$. By Theorem 29.2, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset A$. Then $C = \overline{V}$ is a compact subspace of A containing the neighborhood V of $x \in A$. That is, A is locally compact. □

Corollary 29.4

Corollary 29.4 A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.

Proof. By Theorem 29.1, X is locally compact and Hausdorff if and only if it has a one-point compactification Y, which is compact and Hausdorff. Since $Y \setminus X = \{\infty\}$ and this is a closed set by Theorem 17.8, then X is open in compact Hausdorff space Y. □