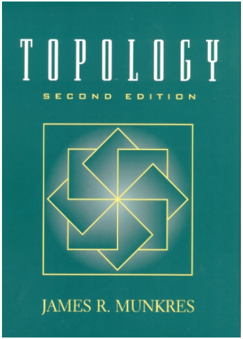
Introduction to Topology

Chapter 3. Connectedness and Compactness

Section 29. Local Compactness—Proofs of Theorems



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(1) X is a subspace of Y.

(2) The set $Y \setminus X$ consists of a single point.

(3) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

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Step 1. We first verify the homeomorphism claim. Let Y and Y' be spaces satisfying the three conditions. Define $Y \to Y'$ by letting h map the "single point" $p \in Y \setminus X$ to the "single point" $q \in Y' \setminus X$, and letting h equal the identity on X. Then h is a bijection (one to one and onto).

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Theorem 29.1 (continued 1)

Proof (continued). Let U be an open set in Y. First, suppose $p \notin U$. Then $h(U) = U$ is open in X (under the subspace topology). Now X is open in Y' since $Y \setminus X$ is closed ($Y' \setminus X$ is a singleton, which forms a closed set, by Theorem 17.8, because Y is compact by (3)) and so U is open in Y'.

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Theorem 29.1 (continued 2)

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Step 2. Suppose X is locally compact and Hausdorff. We construct set Y by adding a single element to X, say $Y = X \cup \{\infty\}$. This give condition (2). Define the collection of subsets of Y, $\mathcal{T} = T_1 \cup T_2$ where
$T_1 = \{U \subset X \mid U \text{ is open in } X\}$ and
$T_2 = \{Y \setminus C \mid C \subset X \text{ is compact in } X\}$.

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$$U_1 \cap U_2 \in T_1$$

$$(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) \in T_2$$

$$U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1) \in T_1.$$

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Proof (continued). Similarly, we have closure under unions:

$$\cup U_\alpha = U \in T_1$$

$$\cup(Y \setminus C_\beta) = Y \setminus (\cap C_\beta) = V \setminus C \in T_2$$

$$(\cup U_\alpha) \cup (\cup Y \setminus C_\beta) = U \cup (Y \setminus C) = T \setminus (C \setminus U) \in T_2.$$

Now we show that X is a subspace of Y (confirming condition (1)). Given any open set U of Y, we need to show that $X \cap U$ is open in X. If $U \in T_1$ then $U \cap X = U$; if $U = Y \setminus C \in T_2$ then $(Y \setminus C) \cap X = X \setminus C \in T_2$.

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Theorem 29.1 (continued 4)

Proof (continued). Now we show that Y is compact. Let $\mathcal{A}$ be an open covering of Y. Since ∞ must be in some element of $\mathcal{A}$, then there is compact $C \subset X$ such that $Y \setminus C \in T_2$ is in $\mathcal{A}$. Since C is compact and $\mathcal{A}$ is a covering of C then there is a finite subcover $\mathcal{A}'$ of $\mathcal{A}$ which covers C.

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Theorem 29.1 (continued 5)

Proof (continued).

Step 3. We now show the converse. Suppose Y satisfies conditions (1), (2), and (3). Then X is Hausdorff because it is a subspace of Y (and it has the subspace topology). Let $x \in X$ be given. Since Y is Hausdorff, there are disjoint open sets U and V in Y containing ∞ and the single point of $Y \setminus X = \{\infty\}$, respectively.

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Theorem 29.2

Theorem 29.2. Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof. If X satisfies this condition, then certainly there is a compact subspace of X (namely $\overline{V}$) containing a neighborhood V of x; that is, the condition implies locally compact.

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Conversely, suppose X is locally compact and let $x \in X$ with U a neighborhood of x. Since S is locally compact, by Theorem 29.1 there is a space Y, the one-point compactification of X. Let $C = Y \setminus U$. Since U is open in X then U is open in Y (in the proof of Theorem 29.1, all sets open in X are open in Y) and so C is closed in Y.

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Proof (continued). Then $\overline{V}$ is compact (again, by Theorem 26.2) and $\overline{V}$ is disjoint from C since $\overline{V} = V \cup V'$ where V' is the set of limit point of set V, and since $x \in V$, $C \subset W$, and $V \cap W = \varnothing$, then no points of C are points of closure of V. So $\overline{V} \subset T \setminus C = U$ is the desired set. □

Corollary 29.3

Corollary 29.3. Let X by locally compact and Hausdorff. Let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Proof. Suppose A is closed in X. Given $x \in A$, let C be a compact subspace of X containing neighborhood U of $x \in X$ (which can be done since X is locally compact).

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Corollary 29.4

Corollary 29.4 A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.

Proof. By Theorem 29.1, X is locally compact and Hausdorff if and only if it has a one-point compactification Y, which is compact and Hausdorff.

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Proof. By Theorem 29.1, X is locally compact and Hausdorff if and only if it has a one-point compactification Y, which is compact and Hausdorff. Since $Y \setminus X = \{\infty\}$ and this is a closed set by Theorem 17.8, then X is open in compact Hausdorff space Y. □

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