Introduction to Topology

Chapter 4. Countability and Separation Axioms

Section 30. The Countability Axioms—Proofs of Theorems

Theorem 30.2

Theorem 30.2. A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Proof. Suppose X is first-countable. Let $A \subset X$ and $a \in A$. Then $a \in X$ and so there is B a countable collection of neighborhoods of a in X such that each neighborhood of x in X contains at least one element of B. Then $\{B \cap A \mid B \in B\}$ is a countable collection of neighborhoods of a in A such that each neighborhood of a in X contains at least one element of B. Then $\{B \cap A \mid B \in B\}$ is a countable collection of neighborhoods of a in A such that each neighborhood of a in A contains at least one element of this collection. So subspace A is first-countable.

Theorem 30.2 (continued)

Proof (continued). Now consider countable collection of first-countable spaces X_i where $i \in \mathbb{N}$. Let $\mathbf{x} = (x_1, x_2, \ldots) \in \prod_{i \in \mathbb{N}} X_i$. Since X_i is first-countable, there is countable collection B_i of neighborhoods of x_i such that each neighborhood of X_i contains at least one element of B_i. Consider the countable collection of products $\prod_{i \in \mathbb{N}} U_i$ where $U_i \in B_i$ for finitely many values of i and $U_i = X_i$ for all other values of i. Then every neighborhood of $\mathbf{x}$ (in the product topology) contains some element of this countable collection and $\prod_{i \in \mathbb{N}} X_i$ if first-countable.

Suppose X is second-countable and that B is a countable basis for X. Then for any $A \subset X$, $\{B \cap A \mid B \in B\}$ is a countable basis for the subspace A and A is second-countable. Now consider countable collection of spaces X_i where $i \in \mathbb{N}$. Suppose B_i is a countable basis for space X_i. Then the countable collection of products $\prod_{i \in \mathbb{N}} U_i$ where $U_i \in B_i$ for finitely many values of i and $U_i = X_i$ for all other values of i, is a basis for $\prod_{i \in \mathbb{N}} X_i$ (under the product topology). So $\prod_{i \in \mathbb{N}} X_i$ is second-countable. □

Theorem 30.3

Theorem 30.3. Suppose X has a countable basis. Then:

(a) Every open covering of X contains a countable subcover.

(b) There exists a countable subset of X that is dense in X.

Proof. Let $\{B_n\}_{n \in \mathbb{N}}$ be a countable basis of X.

(a) Let A be an open covering of X. For each $n \in \mathbb{N}$, choose $A_n \in A$ containing basis element B_n, and let $A' = \{A_n\}_{n \in \mathbb{N}}$. Let $x \in X$. Then $x \in B_n$ for some $n \in \mathbb{N}$ and so $x \in B_n \subset A_n$. So A' is a countable subcover (of X) in A.

(b) For each nonempty basis element B_n, choose a point $x_n \in B_n$. Let $D = \{x_n \mid n \in \mathbb{N}\}$. Let $x \in X$ and let U be an open set containing x. So $x_m \in U$ and x is in $\overline{D}$ (wither $x \in D$ or x is a limit point of D; see Theorem 17.6). That is, $\overline{D} = X$ so that D is dense in X. □