Introduction to Topology

Chapter 4. Countability and Separation Axioms

Section 30. The Countability Axioms—Proofs of Theorems

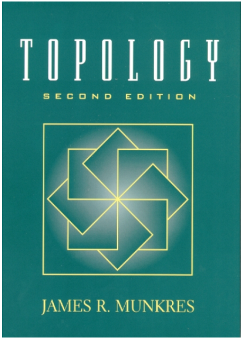

Table of contents

1 Theorem 30.2

2 Theorem 30.3

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Theorem 30.2. A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Proof. Suppose X is first-countable.

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Proof. Suppose X is first-countable. Let $A \subset X$ and $a \in A$. Then $a \in X$ and so there is $\mathcal{B}$ a countable collection of neighborhoods of a in X such that each neighborhood of x in X contains at least one element of $\mathcal{B}$. Then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable collection of neighborhoods of a in A such that each neighborhood of a in X contains at least one element of $\mathcal{B}$.

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Theorem 30.2 (continued)

Proof (continued). Now consider countable collection of first-countable spaces X_i where $i \in \mathbb{N}$. Let $\mathbf{x} = (x_1, x_2, \ldots) \in \prod_{i \in \mathbb{N}} X_i$. Since X_i is first-countable, there is countable collection $\mathcal{B}_i$ of neighborhoods of x_i such that each neighborhood of X_i contains at least one element of $\mathcal{B}_i$. Consider the countable collection of products $\prod_{i \in \mathbb{N}} U_i$ where $U_i \in \mathcal{B}_i$ for finitely many values of i and $U_i = X_i$ for all other values of i.

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Suppose X is second-countable and that $\mathcal{B}$ is a countable basis for X. Then for any $A \subset X$, $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A and A is second-countable.

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Suppose X is second-countable and that $\mathcal{B}$ is a countable basis for X. Then for any $A \subset X$, $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A and A is second-countable. Now consider countable collection of spaces X_i where $i \in \mathbb{N}$. Suppose $\mathcal{B}_i$ is a countable basis for space X_i.

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(a) Every open covering of X contains a countable subcover.

(b) There exists a countable subset of X that is dense in X.

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