Introduction to Topology

Chapter 4. Countability and Separation Axioms

Section 31. The Separability Axioms—Proofs of Theorems

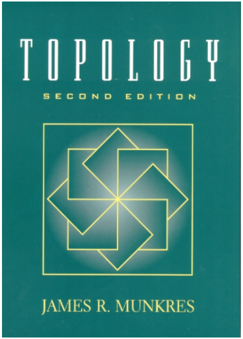

Table of contents

Lemma 31.1

Lemma 31.1. Let X be a topological space. Let one-point sets (singletons) in X be closed.

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of X, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. (a) Let X be regular.

Lemma 31.1

Lemma 31.1. Let X be a topological space. Let one-point sets (singletons) in X be closed.

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of X, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. (a) Let X be regular. Let $x \in X$ and U a neighborhood of x. Let $B = X \setminus U$ so that B is closed. Since X is regular, there are disjoint open sets V and W with $x \in V$ and $B \subset W$.

Lemma 31.1

Lemma 31.1. Let X be a topological space. Let one-point sets (singletons) in X be closed.

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of X, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. (a) Let X be regular. Let $x \in X$ and U a neighborhood of x. Let $B = X \setminus U$ so that B is closed. Since X is regular, there are disjoint open sets V and W with $x \in V$ and $B \subset W$. Now $\overline{V} \cap B = \varnothing$ since $\overline{V} = V \cup V'$ (where V' is the set of limit points of V; see Theorem 17.6) and W is a neighborhood of all points in V which does not intersect V so no point of B is a limit point of V. So open sets V and $X \setminus \overline{V}$ are disjoint with $x \in V$, $B \subset X \setminus \overline{V}$.

Lemma 31.1

Lemma 31.1. Let X be a topological space. Let one-point sets (singletons) in X be closed.

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of X, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. (a) Let X be regular. Let $x \in X$ and U a neighborhood of x. Let $B = X \setminus U$ so that B is closed. Since X is regular, there are disjoint open sets V and W with $x \in V$ and $B \subset W$. Now $\overline{V} \cap B = \varnothing$ since $\overline{V} = V \cup V'$ (where V' is the set of limit points of V; see Theorem 17.6) and W is a neighborhood of all points in V which does not intersect V so no point of B is a limit point of V. So open sets V and $X \setminus \overline{V}$ are disjoint with $x \in V$, $B \subset X \setminus \overline{V}$. Hence X is regular.

Lemma 31.1

Lemma 31.1. Let X be a topological space. Let one-point sets (singletons) in X be closed.

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of X, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. (a) Let X be regular. Let $x \in X$ and U a neighborhood of x. Let $B = X \setminus U$ so that B is closed. Since X is regular, there are disjoint open sets V and W with $x \in V$ and $B \subset W$. Now $\overline{V} \cap B = \varnothing$ since $\overline{V} = V \cup V'$ (where V' is the set of limit points of V; see Theorem 17.6) and W is a neighborhood of all points in V which does not intersect V so no point of B is a limit point of V. So open sets V and $X \setminus \overline{V}$ are disjoint with $x \in V$, $B \subset X \setminus \overline{V}$. Hence X is regular.

Lemma 31.1 (continued)

Lemma 31.1. Let X be a topological space. Let one-point sets (singletons) in X be closed.

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of X, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof (continued). (b) The proof is identical to the proof of (a) with element $x \in X$ replaced with closed set $A \subset X$. □

Lemma 31.1 (continued)

Lemma 31.1. Let X be a topological space. Let one-point sets (singletons) in X be closed.

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of X, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof (continued). (b) The proof is identical to the proof of (a) with element $x \in X$ replaced with closed set $A \subset X$. □

Theorem 31.2

Theorem 31.2.

(a) A subspace of a Hausdorff space is Hausdorff. A product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular. A product of regular spaces is regular.

Proof. (a) Let X be Hausdorff. Let Y be a subspace of X with $x, y \in Y$.

Theorem 31.2

Theorem 31.2.

(a) A subspace of a Hausdorff space is Hausdorff. A product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular. A product of regular spaces is regular.

Proof. (a) Let X be Hausdorff. Let Y be a subspace of X with $x, y \in Y$. If U and V are disjoint neighborhoods of x and y (respectively) in X, then $U \cap Y$ and $V \cap Y$ are disjoint open neighborhoods of x and y (respectively) in Y (under the subspace topology).

Theorem 31.2

Theorem 31.2.

(a) A subspace of a Hausdorff space is Hausdorff. A product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular. A product of regular spaces is regular.

Proof. (a) Let X be Hausdorff. Let Y be a subspace of X with $x, y \in Y$. If U and V are disjoint neighborhoods of x and y (respectively) in X, then $U \cap Y$ and $V \cap Y$ are disjoint open neighborhoods of x and y (respectively) in Y (under the subspace topology).

Let $\{X_\alpha\}$ be a family of Hausdorff spaces. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points in $\prod X_\alpha$.

Theorem 31.2

Theorem 31.2.

(a) A subspace of a Hausdorff space is Hausdorff. A product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular. A product of regular spaces is regular.

Proof. (a) Let X be Hausdorff. Let Y be a subspace of X with $x, y \in Y$. If U and V are disjoint neighborhoods of x and y (respectively) in X, then $U \cap Y$ and $V \cap Y$ are disjoint open neighborhoods of x and y (respectively) in Y (under the subspace topology).

Let $\{X_\alpha\}$ be a family of Hausdorff spaces. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points in $\prod X_\alpha$. Because $\mathbf{x} \neq \mathbf{y}$, there is some β such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff there are disjoint open sets U and V in X_β with $x_\beta \in U$ and $y_\beta \in V$.

Theorem 31.2

Theorem 31.2.

(a) A subspace of a Hausdorff space is Hausdorff. A product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular. A product of regular spaces is regular.

Proof. (a) Let X be Hausdorff. Let Y be a subspace of X with $x, y \in Y$. If U and V are disjoint neighborhoods of x and y (respectively) in X, then $U \cap Y$ and $V \cap Y$ are disjoint open neighborhoods of x and y (respectively) in Y (under the subspace topology).

Let $\{X_\alpha\}$ be a family of Hausdorff spaces. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points in $\prod X_\alpha$. Because $\mathbf{x} \neq \mathbf{y}$, there is some β such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff there are disjoint open sets U and V in X_β with $x_\beta \in U$ and $y_\beta \in V$.

Theorem 31.2 (continued 1)

Proof (continued). Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ where $\mathbf{x} \in \pi_\beta^{-1}(U)$ and $\mathbf{y} \in \pi_\beta^{-1}(V)$. (Recall $\pi_\beta^{-1}(U) = \prod Z_\alpha$ where $Z_\beta = U$ and $Z_\alpha = X_\alpha$ for all $\alpha \neq \beta$.)

(b) Let Y be a subspace of regular space X. Then one-point sets are closed in Y (by definition of regular). Let $x \in X$ and let B be a closed (in Y) subset of Y not containing x. Let $\overline{B}$ denote the closure of B in X.

Theorem 31.2 (continued 1)

Proof (continued). Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ where $\mathbf{x} \in \pi_\beta^{-1}(U)$ and $\mathbf{y} \in \pi_\beta^{-1}(V)$. (Recall $\pi_\beta^{-1}(U) = \prod Z_\alpha$ where $Z_\beta = U$ and $Z_\alpha = X_\alpha$ for all $\alpha \neq \beta$.)

(b) Let Y be a subspace of regular space X. Then one-point sets are closed in Y (by definition of regular). Let $x \in X$ and let B be a closed (in Y) subset of Y not containing x. Let $\overline{B}$ denote the closure of B in X. Then $\overline{B} \cap Y = B$ since B is closed in Y. So $x \not\in \overline{B}$ and since X is regular, there are disjoint open set U and V of X with $s \in U$ and $\overline{B} \subset V$. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y with $x \in U \cap Y$ and $B \subset V \cap Y$. So Y is regular.

Theorem 31.2 (continued 1)

Proof (continued). Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ where $\mathbf{x} \in \pi_\beta^{-1}(U)$ and $\mathbf{y} \in \pi_\beta^{-1}(V)$. (Recall $\pi_\beta^{-1}(U) = \prod Z_\alpha$ where $Z_\beta = U$ and $Z_\alpha = X_\alpha$ for all $\alpha \neq \beta$.)

(b) Let Y be a subspace of regular space X. Then one-point sets are closed in Y (by definition of regular). Let $x \in X$ and let B be a closed (in Y) subset of Y not containing x. Let $\overline{B}$ denote the closure of B in X. Then $\overline{B} \cap Y = B$ since B is closed in Y. So $x \notin \overline{B}$ and since X is regular, there are disjoint open set U and V of X with $s \in U$ and $\overline{B} \subset V$. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y with $x \in U \cap Y$ and $B \subset V \cap Y$. So Y is regular.

Theorem 31.2 (continued 2)

Proof (continued). Let $\{X_\alpha\}$ be a family of regular spaces and let $X = \prod X_\alpha$. Since regular spaces are Hausdorff, part (a) implies that X is Hausdorff, so one-point sets ares closed in X. Let $\mathbf{x} = (x_\alpha) \in X$ and let U be a neighborhood of $\mathbf{x}$ in X. There is a basis element of the product topology, $\prod U_\alpha$, containing $\mathbf{x}$ where $\prod U_\alpha \subset U$. For each α, since X_α is regular, there is a neighborhood V_α of x_α in X_α such that $\overline{V}_\alpha \subset U_\alpha$ by Lemma 31.1(a). If $U_\alpha = X_\alpha$ then set this $V_\alpha = X_\alpha$ (which is the case for all but finitely many α by the definition of product topology).

Theorem 31.2 (continued 2)

Proof (continued). Let $\{X_\alpha\}$ be a family of regular spaces and let $X = \prod X_\alpha$. Since regular spaces are Hausdorff, part (a) implies that X is Hausdorff, so one-point sets ares closed in X. Let $\mathbf{x} = (x_\alpha) \in X$ and let U be a neighborhood of $\mathbf{x}$ in X. There is a basis element of the product topology, $\prod U_\alpha$, containing $\mathbf{x}$ where $\prod U_\alpha \subset U$. For each α, since X_α is regular, there is a neighborhood V_α of x_α in X_α such that $\overline{V}_\alpha \subset U_\alpha$ by Lemma 31.1(a). If $U_\alpha = X_\alpha$ then set this $V_\alpha = X_\alpha$ (which is the case for all but finitely many α by the definition of product topology). Then $V = \prod V_\alpha$ is a neighborhood of x in X (under the product topology). Since $\overline{V} = \prod \overline{V}_\alpha$ by Theorem 19.5, then $\overline{V} \subset \prod U_\alpha \subset U$. So by Lemma 31.1(1), $X = \prod X_\alpha$ is regular. □

Theorem 31.2 (continued 2)

Proof (continued). Let $\{X_\alpha\}$ be a family of regular spaces and let $X = \prod X_\alpha$. Since regular spaces are Hausdorff, part (a) implies that X is Hausdorff, so one-point sets ares closed in X. Let $\mathbf{x} = (x_\alpha) \in X$ and let U be a neighborhood of $\mathbf{x}$ in X. There is a basis element of the product topology, $\prod U_\alpha$, containing $\mathbf{x}$ where $\prod U_\alpha \subset U$. For each α, since X_α is regular, there is a neighborhood V_α of x_α in X_α such that $\overline{V}_\alpha \subset U_\alpha$ by Lemma 31.1(a). If $U_\alpha = X_\alpha$ then set this $V_\alpha = X_\alpha$ (which is the case for all but finitely many α by the definition of product topology). Then $V = \prod V_\alpha$ is a neighborhood of x in X (under the product topology). Since $\overline{V} = \prod \overline{V}_\alpha$ by Theorem 19.5, then $\overline{V} \subset \prod U_\alpha \subset U$. So by Lemma 31.1(1), $X = \prod X_\alpha$ is regular. □