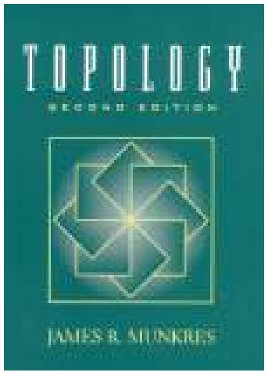
Introduction to Topology

Chapter 4. Countability and Separation Axioms

Section 32. Normal Spaces—Proofs of Theorems



Theorem 32.1

Theorem 32.1. Every regular space with a countable basis is normal.

Proof. Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed sets in X. Since X is regular, each $x \in A$ has a neighborhood U not intersecting B. By Lemma 31.1(a), there is a neighborhood V of x with $\overline{V} \subset U$, and there is a basis element of $\mathcal{B}$ containing x which is a subset of V. Choose such a basis element for each $x \in A$. Then this is a countable (since $\mathcal{B}$ is countable) covering of A by open sets whose closures do not interset B. Denote the sets in this covering as $\{U_n\}_{n\in\mathbb{N}}$.
Similarly, find a countable collection $\{V_n\}$ of open sets covering B such that each set $\overline{V}_n$ is disjoint from A. Then $U = \cup_{n\in\mathbb{N}} U_n$ and $V = \cup_{n\in\mathbb{N}} V_n$ are open sets containing A and B, respectively (but they may not be disjoint). Now, for $n \in \mathbb{N}$, define

$$U_n' = U_n \setminus \cup_{i=1}^n \overline{V}_i \text{ and } V_n' = V_n \setminus \cup_{i=1}^n \overline{U}_i.$$

Then each U_n' and V_n' is open.

Theorem 32.1 (continued)

Theorem 32.1. Every regular space with a countable basis is normal.

Proof (continued). The collection $\{U_n'\}$ covers A and $\{V_n'\}$ covers B (this is where the "$\overline{U}_n$ is disjoint from B" and "$\overline{V}_n$ is disjoint from A" parts are used).

Finally, consider $U' = \cup_{n\in\mathbb{N}} U_n'$ and $V' = \cup_{n\in\mathbb{N}} V_n'$. ASSUME $x \in U' \cap V'$. Then $x \in U_j' \cap V_k'$ for some $j, k \in \mathbb{N}$. If $j \le k$ then $x \in U_j$ (since $U_j' = J_j \setminus \cup_{i=1}^j \overline{V}_i$) but, since $j \le k$, $x \notin V_k$ (since $V_k' = V_k \setminus \cup_{i=1}^k \overline{U}_i$, a CONTRADICTION. A similar contradiction follows if $j \ge k$. So U' and V' are disjoint open sets with $A \subset U'$ and $B \subset V'$. That is, X is regular. □

Theorem 32.2

Theorem 32.2. Every metrizable space is normal.

Proof. Let X be metrizable with metric d. Let A and B be disjoint closed sets in X. For each $a \in A$, choose $\varepsilon_a > 0$ so that $B(a, \varepsilon_a)$ does not intersect B (since B is closed, it contains its limit points by Corollary 17.7, so a is not a limit point of B and such $B(a, \varepsilon_a)$ exists). Similarly, for each $b \in B$ choose $\varepsilon_b > 0$ so that $B(b, \varepsilon_b)$ does not intersect A. Define

$$U = \cup_{a\in A} B(a, \varepsilon_a/2) \text{ and } V = \cup_{b\in B} B(b, \varepsilon_b/2).$$

Then U and V are open sets and $A \subset U$, $B \subset V$. ASSUME $z \in U \cap V$. Then $z \in B(a, \varepsilon_a/2)$ and $z \in B(d, \varepsilon_b/2)$ for some $a \in A$ and $b \in B$. By the Triangle Inequality,

$$d(a,b) \le d(a,z) + d(z,b) < \varepsilon_a/2 + \varepsilon_b/2.$$

If $\varepsilon_a \le \varepsilon_b$ then $d(a,b) < \varepsilon_b$ and then $a \in B(b, \varepsilon_b)$, a CONTRADICTION.

Theorem 32.2 (continued)

Theorem 32.2. Every metrizable space is normal.

Proof (continued) . Similarly, if $\varepsilon_b \leq \varepsilon_a$ then $d(a,b) < \varepsilon_a$ and $b \in B(a, \varepsilon_a)$, a contradiction. So the assumption that such $z \in U \cap V$ exists is false and U and V are disjoint open sets with $A \subset U$ and $B \subset V$. Therefore, X is normal. □

Theorem 32.3

Theorem 32.3. Every compact Hausdorff space is normal.

Proof. Let X be a compact Hausdorff space. Let A and B be disjoint closed sets in X. By Lemma 26.4, for each $a \in A$, there are disjoint open U_a and V_a with $x \in U_x$ and $B \subset V_x$. Since A is closed and X is Hausdorff, then A is compact by Theorem 26.2, so the open covering $\{U_a\}_{a \in A}$ of A has a finite subcover, say $\{U_1, U_2, \ldots, U_n\}$. Then $U = U_1 \cap U_2 \cap \cdots \cap U_n$ and $V = V_1 \cap V_2 \cap \cdots \cap V_n$ are disjoint open sets where $A \subset U$ and $B \subset V$. That is, X is regular. □

Theorem 32.4

Theorem 32.4. Every well-ordered set X is normal in the order topology.

Proof. Let X be a well-ordered set. We claim that every interval of the form $(x, y]$ is open in X. If X has a largest element and y is that element, then $(x, y]$ is a basis element of y (see the definition of "order topology" in Section 14). If y is not the largest element of X, then $(x, y]$ equals the open set (x, y') where y' is the immediate successor of y (since X is well-ordered, every nonempty subset of X has a smallest element and so every element $x \in X$ other than the largest element of X has an immediate successor; namely the smallest element of $\{y \in X \mid v > x\}$). In either case, $(x, y]$ is open in X.

Now let A and B be disjoint closed sets in X. First, suppose that neither A nor B contains the smallest element a_0 of X. For each $a \in A$, there is a basis element containing a disjoint from B (since B is closed it contains its limit points by Corollary 17.7, so a is not a limit point of B).

Theorem 32.4 (continued 1)

Proof (continued). Since a is not the smallest element of X, the basis element containing a contains some interval of the form $(x, a]$. For each $a \in A$, choose such an interval $(x_a, a]$ disjoint from set B. Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from set A. Notice that each $(x_a, a]$ and $(y_b, b]$ is open since each is of the form $(x_a, a]$ and $(y_b, b]$ is open since each is of the form $(x_a, a+1)$ and $(y_b, b+1)$ where "+1" represents the immediate successor. The sets

$$U = \cup_{a \in A}(x_a, a] \text{ and } V = \cup_{b \in B}(y_b, b]$$

are open sets where $A \subset U$ and $B \subset V$. ASSUME $z \in U \cap V$. Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$. WLOG, $a < b$. If $a \leq y_b$ then the two intervals are disjoint CONTRADICTING the assumption that $z \in (z_a, a] \cap (y_b, b]$. If $a > y_b$ then $y_b < a < b$ and $a \in (y_b, b]$, CONTRADICTING the fact that $(y_b, b]$ is disjoint from A. So the assumption that there is $z \in U \cap V$ is false and so U and V are in fact disjoint.

Theorem 32.4 (continued 2)

Theorem 32.4. Every well-ordered set X is normal in the order topology.

Proof (continued). So the normality condition is satisfied when neither (closed) A nor B contains the smallest element of X.

Finally, suppose A and B are disjoint closed sets in X where A contains the smallest element a_0 in X where A contains the smallest element a_0 of X. The set $\{a_0\}$ is both open and closed in X, $\{a_0\} = [a_0, a_0 + 1)$ and $X \setminus \{a_0\} = \cup_{x \in X}(a_0, x)$. By the previous paragraph, there exist disjoint open sets U and V, neither containing a_0, where $A \setminus \{a_0\} \subset U$ and $B \subset V$ (where $A \setminus \{a_0\}$ and B are closed, disjoint sets). Then $U \cup \{a_0\}$ and V are disjoint open sets containing A and B respectively. So the normality condition is satisfied when one of A or B contains the smallest element of X. Hence, X is normal. □