Introduction to Topology

Chapter 4. Countability and Separation Axioms

Section 32. Normal Spaces—Proofs of Theorems

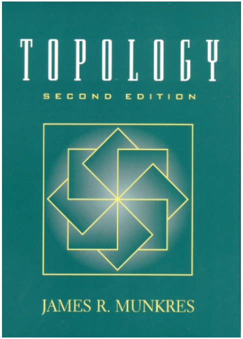

Table of contents

Theorem 32.1

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Proof. Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed sets in X. Since X is regular, each $x \in A$ has a neighborhood U not intersecting B. By Lemma 31.1(a), there is a neighborhood V of x with $\overline{V} \subset U$, and there is a basis element of $\mathcal{B}$ containing x which is a subset of V.

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Similarly, find a countable collection $\{V_n\}$ of open sets covering B such that each set $\overline{V}_n$ is disjoint from A. Then $U = \cup_{n \in \mathbb{N}} U_n$ and $V = \cup_{n \in \mathbb{N}} V_n$ are open sets containing A and B, respectively (but they may not be disjoint).

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$$U_n' = U_n \setminus \cup_{i=1}^{n} \overline{V}_i \text{ and } V_n' = V_n \setminus \cup_{i=1}^{n} \overline{U}_i.$$

Then each U_n' and V_n' is open.

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Theorem 32.1 (continued)

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Proof (continued). The collection $\{U'_n\}$ covers A and $\{V'_n\}$ covers B (this is where the "$\overline{U}_n$ is disjoint from B" and "$\overline{V}_n$ is disjoint from A" parts are used).

Finally, consider $U' = \cup_{n \in \mathbb{N}} U'_n$ and $V' = \cup_{n \in \mathbb{N}} V'_n$. ASSUME $x \in U' \cap V'$. Then $x \in U'_j \cap V'_k$ for some $j, k \in \mathbb{N}$.

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$$U = \cup_{a \in A} B(a, \varepsilon_a/2) \text{ and } V = \cup_{b \in B} B(b, \varepsilon_b/2).$$

Then U and V are open sets and $A \subset U$, $B \subset V$.

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Then U and V are open sets and $A \subset U$, $B \subset V$. ASSUME $z \in U \cap V$. Then $z \in B(a, \varepsilon_a/2)$ and $z \in B(d, \varepsilon_b/2)$ for some $a \in A$ and $b \in B$. By the Triangle Inequality,

$$d(a, b) \leq d(a, z) + d(z, b) < \varepsilon_a/2 + \varepsilon_b/2.$$

If $\varepsilon_a \leq \varepsilon_b$ then $d(a, b) < \varepsilon_b$ and then $a \in B(b, \varepsilon_b)$, a CONTRADICTION.

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Theorem 32.2 (continued)

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Proof (continued) . Similarly, if $\varepsilon_b \leq \varepsilon_a$ then $d(a, b) < \varepsilon_a$ and $b \in B(a, \varepsilon_a)$, a contradiction. So the assumption that such $z \in U \cap V$ exists is false and U and V are disjoint open sets with $A \subset U$ and $B \subset V$. Therefore, X is normal. □

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Theorem 32.4

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Proof. Let X be a well-ordered set. We claim that every interval of the form $(x, y]$ is open in X.

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Proof. Let X be a well-ordered set. We claim that every interval of the form $(x, y]$ is open in X. If X has a largest element and y is that element, then $(x, y]$ is a basis element of y (see the definition of "order topology" in Section 14).

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Theorem 32.4 (continued 1)

Proof (continued). Since a is not the smallest element of X, the basis element containing a contains some interval of the form $(x, a]$. For each $a \in A$, choose such an interval $(x_a, a]$ disjoint from set B. Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from set A. Notice that each $(x_a, a]$ and $(y_b, b]$ is open since each is of the form $(x_a, a]$ and $(y_b, b]$ is open since each is of the form $(x_a, a+1)$ and $(y_b, b+1)$ where "$+1$" represents the immediate successor.

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$$U = \cup_{a \in A}(x_a, a] \text{ and } V = \cup_{b \in B}(y_b, b]$$

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Theorem 32.4. Every well-ordered set X is normal in the order topology.

Proof (continued). So the normality condition is satisfied when neither (closed) A nor B contains the smallest element of X.

Finally, suppose A and B are disjoint closed sets in X where A contains the smallest element a_0 in X where A contains the smallest element a_0 of X. The set $\{a_0\}$ is both open and closed in X, $\{a_0\} = [a_0, a_0 + 1)$ and $X \setminus \{a_0\} = \cup_{x \in X}(a_0, x)$.

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