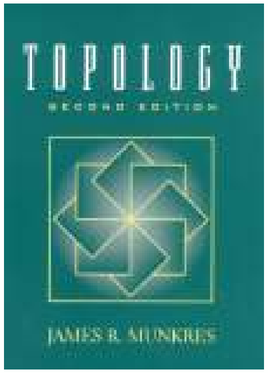
Introduction to Topology

Chapter 4. Countability and Separation Axioms
Section 33. The Urysohn Lemma—Proofs of Theorems



Theorem 33.1. The Urysohn Lemma

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Let X be a normal space. Let A and B be disjoint closed subsets of X. Let $[a,b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a,b]$ such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.

Proof. Without loss of generality, we take $[a,b] = [0,1]$. We use normality to construct a nested family of open sets which is indexed by the rational numbers in $[0,1]$. Continuous f is then defined using the sets. We follow Munkres' four steps.
Step 1. let $P = [0,1] \cap \mathbb{Q}$. Since P is countable, there is mapping $N : \mathbb{Q} \to \mathbb{N}$ which is a bijection. Take $N(1) = 1$ and $N(2) = 0$. We now define indexed sets $U_{N(p)}$ for $p \in P$. First, define $U_{N(1)} = U_1 = X \setminus B$. Second, because A is a closed set contained in open set U_1, by the normality of X, Lemma 31.1(b) implies that there is open $U_{N(2)} = U_0$ such that $A \subset U_0$ and $\overline{U}_0 \subset U_1$.

Theorem 33.1 (continued 1)

Proof (continued). In general, let $P_n = \{N(1), N(2, \ldots, N(n)\}$ and suppose for $p, q \in P_n$ with $p < q$, we have already defined open U_p, U_q with $\overline{U}_p \subset U_q$. We now inductively define $U_{N(n+1)}$. Let $N(n+1) = r$ and $P_{n+1} = P_n \cup \{N(n+1)\} = P \cup \{r\}$. Now $r \neq 0, 1$ and so r has an immediate predecessor in P_{n+1}, say p, and an immediate successor in P_{n+1}, say q (this follows from Theorem 10.1). Since $p, q \in P_n$ then we are supposing that U_p and U_q are already defined with $\overline{U}_p \subset U_q$. By the normality of X, there is open $U_r \subset X$ such that $\overline{U}_p \subset U_r$ and $\overline{U}_r \subset U_q$ by Lemma 31.1(b). In this way, we have $U_{N(n)}$ defined for all $n \in \mathbb{N}$; that is, for all $p \in P$ we have defined open U_p. We claim that $p < q$ implies $\overline{U}_p \subset U_q$ for all $p, q \in P$. Let $p, q \in P_{n+1}$ with $p < q$. If $p, q \in P_n$ then $\overline{U}_p \subset U_q$ by the induction hypothesis. If one of p and q is r and the other is $s \in P_n$, then either $s \leq p$ in which case $\overline{U}_s \subset \overline{U}_p \subset U_r$ or $s \geq q$ in which case $\overline{U}_r \subset U_q \subset U_s$. Therefore, by induction, for any $p, q \in P$ we have $p < q$ implies $\overline{U}_p \subset U_q$. The sets are as illustrated in Figure 33.1.

Theorem 33.1 (continued 2)

Proof (continued).

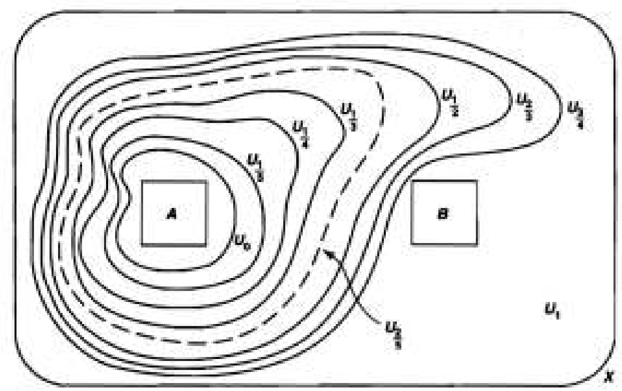

Figure 33.1

Step 2. In this step, we extend the definition to U_p from $p \in [0,1] \cap \mathbb{Q}$ to all of $\mathbb{Q}$ by setting $U_p = \varnothing$ if $p < 0$ and $U_p = X$ if $p > 1$. We still have $p < q$ implying $\overline{U}_p \subset U_q$ for all $p, q \in \mathbb{Q}$.

Theorem 33.1 (continued 3)

Proof (continued).
Step 3. We now define f. For $x \in X$ define $\mathbb{Q}(x) = \{p \in \mathbb{Q} \mid x \in U_p\}$ where U_p is as defined above. Since $U_p = \varnothing$ for $p < 0$, for all $x \in X$ the set $\mathbb{Q}(x)$ contains no rationals less than 0. Since $U_p = X$ for $p > 1$, then all $x \in X$ are in U_p for $p > 1$. So $\mathbb{Q}(x)$ is bounded below and so has a greatest lower bound in $[0,1]$. Define

$$f(x) = \inf \mathbb{Q}(x) = \inf\{p \in \mathbb{Q} \mid x \in U_p\}.$$

Step 4. If $x \in A$ then $x \in U_p$ for every rational $p \geq 0$ (since $A \subset U_p$ for all $p \geq 0$) and so $f(x) = 0$ for all $x \in A$, as desired. If $x \in B$, then $x \in U_p$ for no rational $p \leq 1$ but $x \in U_p = X$ for all rational $p > 1$. Hence $f(a) = 1$ for all $x \in B$, as desired.

Theorem 33.1 (continued 4)

Proof (continued). Now to show f is continuous. We first prove two things:

(1) If $x \in \overline{U}_r$ then $f(x) \leq r$.

(2) If $x \notin U_r$ then $f(x) \geq r$.

To prove (1), note that if $x \in \overline{U}_r$ then $x \in U_s$ for every $s > r$ (by the construction of the U_p in Step 1). Therefore $\mathbb{Q}(x)$ contains all rationals greater than r and so $f(x) = \inf \mathbb{Q}(x) \leq r$ for $x \in \overline{U}_r$. To prove (2), not that if $x \notin U_r$ then $x \notin U_s$ for any $s < r$ (since $\overline{U}_s \subset U_r$ for $s < r$). So $\mathbb{Q}(x)$ contains no rational numbers less than r, so that $f(x) = \inf \mathbb{Q}(x) \geq r$ for $x \notin U_r$.
Now given $x_0 \in X$ and open interval $(c,d) \subset \mathbb{R}$ containing $f(x_0)$, choose rational p and q such that $c < p < f(x_0) < q < d$. Consider $U = U_q \setminus \overline{Q}_p$. We have $f(x_0) < q$ so by the contrapositive of condition (2) we have that $x_0 \in U_q$. Since $f(x_0) > p$, the contrapositive of (1) implies that $x_0 \notin \overline{U}_p$. So $x_0 \in U = U_q \setminus \overline{U}_p$.

Theorem 33.1 (continued 5)

Theorem 33.1. The Urysohn Lemma.
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Proof (continued). Let $x \in U$. Then $x \in U_q \subset \overline{U}_q$ so that $f(x) \leq q$ by condition (1). Since $x_0 \notin \overline{U}_p$ then $x_0 \notin U_p$ and $f(x) \geq p$ by condition (2). Therefore, $f(x) \in [p.q] \subset (c,d)$. So $f(U) \subset (c,d)$ and $U = U_q \setminus \overline{U}_p - U_q \cap (X \setminus \overline{U}_p)$ is an open set containing x_0 such that $f(U) \subset (c,d)$. So f is continuous at arbitrary point $x_0 \in X$ and f is the desired function. □

Theorem 33.2

Theorem 33.2. A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

Proof. Let X be completely regular and let Y be a subspace of X. Let $x_0 \in Y$ and let A be a closed set of Y not containing x_0. Since A is closed in Y, $A = \overline{A} \cap Y$ where $\overline{A}$ denotes the closure of A in X. So $x_0 \notin \overline{A}$. Since X is completely regular, by definition there is continuous $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(\overline{A}) = \{0\}$. The restriction of f to Y is the desired function showing that Y is completely regular.
Let $X = \prod X_\alpha$ be a product of completely regular spaces. Let $\mathbf{b} = (b_\alpha)$ be a point of X and let A be a closed set of X not containing $\mathbf{b}$. Choose a basis element $\prod U_\alpha$ containing $\mathbf{b}$ that does not intersect A (which can be done since A is closed and so $\mathbf{b}$ is not a limit point of A). Then (under the product topology) $U_\alpha = X_\alpha$ except for finitely many α, say $\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Theorem 33.2 (continued)

Proof (continued). Given $i = 1, 2, \ldots, n$, choose continuous $f : X_{\alpha_i} \longrightarrow [0,1]$ such that $f_i(b_{\alpha_i}) = 1$ and $f_i(X_{\alpha_i} \setminus U_{\alpha_i}) = \{0\}$ (using the complete regularity of each X_{α_i}). Let $\varphi_i : X \longrightarrow [0,1]$ as $\varphi_i(\mathbf{x}) = f_i(\pi_{\alpha_i}(\mathbf{x}))$ (where π_{α_i} is the projection of X into X_{α_i}). The projection π_{α_i}) is continuous (see the proof of Theorem 19.6) and so each φ_i is continuous. Also, for $\mathbf{x} \notin \pi_{\alpha_i}^{-1}(U_{\alpha_i})$, $\varphi_i(\mathbf{x}) = f_i(\pi_{\alpha_i}(\mathbf{x})) = f_i(x_{\alpha_i}) = 0$ since $x_{\alpha_i} \in X_{\alpha_i} \setminus U_{\alpha_i}$. So φ_i is zero on $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$; in particular, φ_i is zero on A. Then the product $f(\mathbf{x}) = \varphi_i(\mathbf{x})\varphi_2(\mathbf{x}) \cdots \varphi_n(\mathbf{x})$ is continuous and for $\mathbf{x} \in A$, we have $\varphi_i(\mathbf{x}) = 0$. Also,

$$f(\mathbf{b}) = \varphi_i(\mathbf{b})\varphi_2(\mathbf{b}) \cdots \varphi_n(\mathbf{b}) = f_1(\pi_{\alpha_1}(\mathbf{b}))f_2(\pi_{\alpha_2}(\mathbf{b})) \cdots f_n(\pi_{\alpha_n}(\mathbf{b}))$$

$$= f_1(b_1)f_2(b_2) \cdots f_n(b_n) = (1)(1) \cdots (1) = 1.$$

So f is the desired continuous function and shows that $\prod X_\alpha$ is complete regular. □