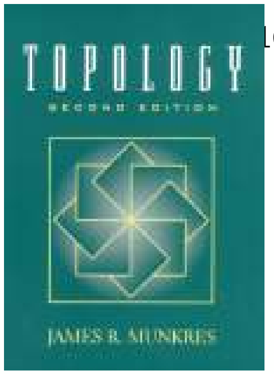
Introduction to Topology

Chapter 4. Countability and Separation Axioms

Section 34. The Urysohn Metrization Theorem—Proofs of Theorems



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Lemma 34.A

Lemma 34.A. If X is a regular space with a countable basis, then there exists a countable collection of continuous functions $f_m : X \to [0.1]$ having the property that given any point $x_0 \in X$ and any neighborhood U of x_0, there exists an index n such that $f_n(x_0) > 0$ and $f_n(x) = 0$ for all $x \in U$.

Proof. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable basis for X. Since X is regular then it is Hausdorff and so (by Theorem 17.8) $\{x\}$ is a closed set for all $x \in X$. So for any basis element B_n and for each $x \in B_n$, there is an open set (and hence a basis element) B_m such that $x \in B_m \subset \overline{B}_m \subset B_n$ by Lemma 31.1(b). So for each $n, m \in \mathbb{N}$ for which $\overline{B}_n \subset B_m$, by the Urysohn Lemma there is a continuous function $g_{n,m} : X \to [0,1]$ such that $g_{n,m}(\overline{B}_n) = \{1\}$ and $g_{n,m}(X \setminus B_m) = \{0\}$. Given any arbitrary $x_0 \in X$ and neighborhood U of x_0, there is a basis element B_m containing x_0 that is contained in U. By Lemma 31.1(b) (as above) there is B_n so that $x_0 \in B_n$ and $\overline{B}_n \subset B_m$.

Lemma 34.A (continued)

Lemma 34.A. If X is a regular space with a countable basis, then there exists a countable collection of continuous functions $f_m : X \to [0,1]$ having the property that given any point $x_0 \in X$ and any neighborhood U of x_0, there exists an index n such that $f_n(x_0) > 0$ and $f_n(x) = 0$ for all $x \in U$.

Proof (continued). Then pair $(n,m) \in \mathbb{N}\times\mathbb{N}$ is such that $g_{n,m}$ is defined, $g_{n,m}(x_0) = 1 > 0$ (since $x_0 \in B_m \subset \overline{B}_m$) and for $x \in X \setminus U$ (i.e., $x \notin U$) we have $g_{n,m}(x) = 0$ since $x \in X \setminus U \subset X \setminus B_m$. So $g_{n,m}$ satisfies the required conditions for given x_0 and U. Since x_0 and U are arbitrary and the set of indices $(n,m) \in \mathbb{N}\times\mathbb{N}$ for which $g_{n,m}$ is defined on a subset of $\mathbb{N}\times\mathbb{N}$, and so the collection of $g_{n,m}$ is countable (and can be relabeled and indexed as $\{f_n\}_{n\in\mathbb{N}}$). □

Theorem 34.1. The Urysohn Metrization Theorem, First Proof

Theorem 34.1. The Urysohn Metrization Theorem.
Every regular space X with a countable basis is metrizable.

Proof. Let $\mathbb{R}^\omega$ have the product topology (where the basis consists of sets of the form $\prod_{n\in\mathbb{N}} U_n$ where U_n is open in $\mathbb{R}$ and $U_n = \mathbb{R}$ for all but finitely many $n \in \mathbb{N}$). By Lemma 34.A, there are the functions $\{f_n\}_{n\in\mathbb{N}}$ as described. Define $F : X \to \mathbb{R}^\omega$ as $F(x) = (f_1(x), f_2(x), \ldots)$. We claim that F is an embedding (that is, F is a homeomorphism with its image). Since each f_n is continuous and $\mathbb{R}^\omega$ is under the product topology, then F is continuous by Theorem 19.6. If $x \neq y$ then, since X is regular there is open U containing x and not containing y. So for some f_n we have $f_n(x) > 0$ and $f_n(y) = 0$. So $F(x) \neq F(y)$ and F is ont to one (injective). Let $Z = F(X)$. Since F is one to one, then F is a bijection from X to Z.

Theorem 34.1. The Urysohn Metrization Theorem, First Proof (continued 1)

Proof (continued). To show F is a homeomorphism, let U be an open subset of X. Let $z_0 \in F(U)$ and $x_0 \in U$ with $F(x_0) = z_0$. By Lemma 34.A, there is $N \in \mathbb{N}$ for which $f_N(x_0) > 0$ and $f_N(X \setminus U) = \{0\}$. Define open set $V = \pi_N^{-1}((0,\infty)) \subset \mathbb{R}^\omega$ (open since the projection mappings are continuous). Define $W = V \cap Z$ and so W is open in Z (by the definition of the subspace topology).

We now show that $z_0 \in W \subset F(U)$. First, $z_0 \in W$ because

$$\begin{aligned} \pi_N(z_0) &= \pi_N(F(x_0)) \text{ since } z_0 = F(x_0) \\ &= f_N(x_0) \text{ since } F(x) = (f_1(x), f_2(x), \ldots) \\ &> 0 \text{ by the choice of } N \in \mathbb{N}. \end{aligned}$$

Second, if $a \in W$ then $z \in Z = F(X)$ and so $z = F(x)$ for some $x \in X$, and $\pi_N(x) \in (0,\infty)$ since $x \in V \subset W$. Since $\pi_N(z) = \pi_N(F(z)) = f_N(z)$, and f_N equals 0 outside of U, the point x must be in U.

Theorem 34.1. The Urysohn Metrization Theorem, First Proof (continued 2)

Theorem 34.1. The Urysohn Metrization Theorem.
Every regular space X with a countable basis is metrizable.

Proof (continued). That is, $z = F(x) \in F(U)$. Since z is an arbitrary element of W, then $W \subset F(U)$. Since z_0 is an arbitrary element of $F(U)$ and W is an open subset in $Z = F(X)$ containing z_0, then $F(U)$ is open in $F(X)$. Since U is an arbitrary open subset of X and $F(U)$ is open in $F(X)$, then F maps open sets to open sets; that is, F^{-1} is continuous. Therefore F is a continuous bijection with a continuous inverse from X to $[0,1]^\omega \subset \mathbb{R}^\omega$. That is, F is a homeomorphism between X and $[0,1]^\omega$ (and so F is an embedding of X into $\mathbb{R}^\omega$). Now $\mathbb{R}^\omega$ is metrizable by Theorem 20.5, so the subspace $[0,1]^\omega$ is metrizable and hence X is metrizable. □

Theorem 34.1. The Urysohn Metrization Theorem, Second Proof

Theorem 34.1. The Urysohn Metrization Theorem.
Every regular space X with a countable basis is metrizable.

Proof. In this second proof, we embed X in the metric space $(\mathbb{R}, \overline{\rho})$ where $\overline{\rho}(\mathbf{x},\mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in \mathbb{N}\}$, where $\overline{d}(x,y) = \min\{d(x,y), 1\}$ for $x, y \in \mathbb{R}$ (see Section 20). Actually, we embed X in the subspace $[0,1]^\omega$ on which the metric satisfies $\rho(\mathbf{x},\mathbf{y}) = \overline{\rho}(\mathbf{x},\mathbf{y}) = \sup\{|x_i - y_i| \mid i \in \mathbb{N}\}$. We slightly modify the countable collection of functions $f_n : X \to [0,1]$ of Lemma 34.A by replacing f_n by f_n/n so that $f_n(x) \le 1/n$ for all $x \in X$. Define $F : X \to [0,1]^\omega$ as $F(x) = (f_1(x), f-2(x), \ldots)$, as in the first proof. From the first proof, we know that F is one to one. Also from the first proof, under the product topology on $[0,1]^\omega$, the map F carries open sets of X onto open sets of the subspace $Z = F(X)$. The metric ρ is the same as the uniform metric $\overline{\rho}$ on $[0,1]^\omega$, so $[0,1]^\omega$ has the subspace topology as a subspace of $\mathbb{R}^\omega$ which has the uniform (metric) topology.

Theorem 34.1. The Urysohn Metrization Theorem, Second Proof (continued 1)

Proof (continued). By Theorem 20.4, the uniform topology on $\mathbb{R}^\omega$ is finer than the product topology, so the topology on $[0,1]^\omega$ which we have here is finer than the product topology on $[0,1]^\omega$. Therefore, $F : X \to [0,1]^\omega$ also carries open sets of X onto open sets of $[0,1]^\omega$ under the metric topology induced by ρ (since the metric topology has more open sets than the product topology). That is, F^{-1} is continuous. Next, we show that F is continuous.

Let $x_0 \in X$ and $\varepsilon > 0$. First, there is $N \in \mathbb{N}$ such that $1/N < \varepsilon/2$. Since each f_h is continuous (Lemma 34.A), then for $n = 1, 2, \ldots, N$ there is a neighborhood $U_n \subset X$ of x_0 such that $|f_n(x) - f_n(x_0)| \le \varepsilon/2$ for all $x \in U_n$. Let $U = U_1 \cap U_2 \cap \cdots \cap U_N$. Now let $x \in U$. If $n \le N$ then $|f_n(x) - f_n(x_0)| < \varepsilon/2$ by the choice of U and if $n > N$ then $|f_n(x) - f_n(x_0)| < 1/N \le \varepsilon/2$ since we required $f_n(x) \le 1/n$ and so $f_n(x), f_n(x_0) \in [0, 1/n]$.

Theorem 34.1. The Urysohn Metrization Theorem, Second Proof (continued 2)

Theorem 34.1. The Urysohn Metrization Theorem.
Every regular space X with a countable basis is metrizable.

Proof (continued). Therefore

$$\rho(F(x), F(x_0)) = \sup\{|f_n(x) - f_n(x_0)| \mid n \in \mathbb{N}\} \leq \varepsilon/2 < \varepsilon.$$

That is, for any given $x_0 \in X$, for all $\varepsilon > 0$ there is open $U \subset X$ containing x_0 such that if $x \in U$ then $\rho(F(x), F(x_0)) < \varepsilon$. That is, F is continuous. So F is one to one, F is continuous, and F^{-1} is continuous. That is, F is a homeomorphism with $F(X) \subset [0,1]^\omega$ and so F embeds X in $[0,1]^\omega$ (where $[0,1]^\omega$ is a subspace of the metric space $(\mathbb{R}^\omega, \overline{\rho})$, and so it itself a metric space). So X is metrizable as claimed. □

Theorem 34.3

Theorem 34.3. A space X is completely regular if and only if it is homeomorphic to a subspace of $[0,1]^J$ for some indexing set J.

Proof. If X is completely regular then (by definition) one-point sets are closed and there is a family of continuous functions each mapping X to $[0,1]$ which separate points from closed sets. So by the Embedding Theorem (Theorem 34.2), there is an embedding of X in $[0,1]^J$.

If X is homeomorphic to a subspace of $[0,1]^J$. Since $[0,1]^J$ is a metric space, it is Hausdorff and so by Theorem 17.8 each one-point set is closed. By Exercise 33.9, $\mathbb{R}^J$ under the box topology is completely regular. Since the box topology is finer than the product topology, each $f : X \to [0,1]$ in the definition of completely regular which is continuous in the box topology is also continuous in the product topology. Therefore$\mathbb{R}^J$ under the product topology is completely regular and hence X is completely regular. □