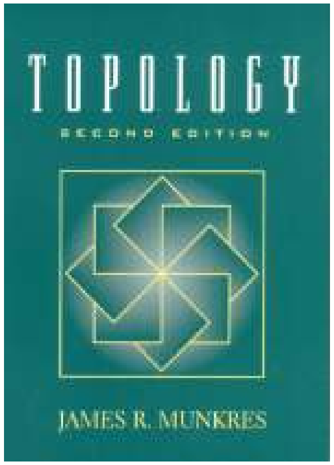
Introduction to Topology

Chapter 4. Countability and Separation Axioms

Section 35. The Tietze Extension Theorem—Proofs of Theorems



Theorem 35.1. The Tietze Extension Theorem

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Let X be a normal space. Let A be a closed subspace of X.

(a) Any continuous function of A into the closed interval $[a,b] \subset \mathbb{R}$ may be extended to a continuous function on all of X into $[a,b]$.

(b) Any continuous function of A into $\mathbb{R}$ may be extended to a continuous function on all of X into $\mathbb{R}$.

Proof. We follow Munkres' three step proof and construct the desired function as a limit of a sequence of functions.
Step 1. First, we consider the case $f : A \to [-r, r]$. We construct continuous $g : X \to \mathbb{R}$ such that $|g(x)| < r/3$ for all $x \in X$ and $|g(a) - f(a)| < 2r/3$ for all $a \in A$. For the construction, define $I_1 = [-r, -r/3]$, $I_2 = [-r/3, r/3]$, and $I_3 = [r/3, r]$. Let $B = f^{-1}(I_1)$ and $C = f^{-1}(I_3)$ (subsets of A. Since f is continuous then B and C are closed (by Theorem 18.1(3)) and are disjoint.

Theorem 35.1. The Tietze Extension Theorem (continued)

Proof (continued). Therefore B and C are closed in X (by Theorem 17.2). By Urysohn's lemma (Theorem 33.1) there is a continuous $g : X \to [-r/3, r/3]$ such that $g(x) = -r/3$ for $x \in B$ and $g(x) = r/3$ for $x \in C$.

Then $|g(x)| \le r/3$ for all $x \in X$. To show $|g(a) - f(a)| < 2r/3$ for all $a \in A$, we consider three cases. First, if $a \in B$ then $f(a) \in I_1$ (since $B = f^{-1}(I_1)$) and $g(a) = -r/3 \in I_1 = [-r/3, r/3]$. Second, if $a \in C$ then $f(a) \in I_3$ (since $C = f^{-1}(I_3)$) and $g(a) = r/3 \in I_3 = [r/3, r]$. Thirdly, if $a \in B \cup C$ then $f(a) \in (-r/3, r/3) \subset [-r/3, r/3] = I_2$ (since a is not in $B = f^{-1}([-r, -r/3])$ nor in $C = f^{-1}([r/3, r])$) and $g(a) \in I_3 = [-r/3, r/3]$. In each case, $|g(a) - f(a)| \le 2r/3$ since

$$\mathrm{diam}(I_1) = \mathrm{diam}(I_2) = \mathrm{diam}(I_3) = 2r/3.$$

Theorem 35.1. Tietze Extension Theorem (continued 1)

Proof (continued).
Step 2. We now prove part (a). Without loss of generality we take $[a,b] = [-1,1]$. Let $f : X \to [-1,1]$ be a continuous function. With $r = 1$, by Step 1 there is continuous real-valued g_1 defined on all of X such that

$$\begin{aligned} |g_1(x)| &\le 1/3 \text{ for } x \in X \\ |f(a) - g_1(a)| &\le 2/3 \text{ for } a \in A. \end{aligned}$$

Now consider the function $f - g_1$. This maps A into the interval $[-2/3, 2/3]$ (since $|f(a) - g_1(a)| \le 2/3$ for all $a \in A$). Now let $r = 2/3$ and apply Step 1 to $f - g_1$ to product real-valued functions g_2 defined on all of X such that

$$\begin{aligned} |g_2(x)| &\le (1/3)(2/3) \text{ for } x \in X \\ |(f(a) - g_1(a))) - g_2(a)| &\le (2/3)^2 \text{ for } a \in A. \end{aligned}$$

So $f - g_1 - g_2$ maps A into the interval $[-(2/3)^2, (2/3)^2]$.

Theorem 35.1. Tietze Extension Theorem (continued 2)

Proof (continued). Apply Step 1 to $f - g_1 - g_2$ to produce g_3, and so forth so that $g_1, g_2, \ldots, g_n$ are defined on X and

$$|f(a) - g_1(a) - g_2(a) - \cdots - g_n(a)| \le (2/3)^n$$

for all $a \in A$. Inductively, Step 1 gives g_{n+1} defined on X such that

$$\begin{aligned} |g_{n+1}(x)| &\le (1/3)(2/3)^n \text{ for } x \in X \\ |f(a) - g_1(a) - g_2(a) - \cdots - g_{n+1}(a)| &\le (2/3)^{n+1} \text{ for } a \in A. \end{aligned}$$

So such g_n exist for all $n \in \mathbb{N}$.
Define $g(x) - \sum_{n+1}^{\infty} g_n(x)$ for $x \in X$. For any $x \in X$,

$$\sum_{n=1}^{\infty} |g_n(x)| \le \sum_{n=1}^{\infty} (1/3)(2/3)^{n-1} = \left(\frac{1}{3}\right)\left(\frac{1}{1-2/3}\right) = 1$$

and so the series converges absolutely for all $x \in X$.

Theorem 35.1. Tietze Extension Theorem (continued 3)

Proof (continued). To show g is continuous, we need to show the convergence of the sequence of partial sums if *uniform* (recall that the uniform limit of a sequence of continuous functions is continuous by the Uniform Limit Theorem, Theorem 21.6, since $\mathbb{R}$ is a metric space). Recall the Weierstrass M-Test (see page 135): If $f_i : X \to \mathbb{R}$ with $|f_i(x)| \le M_i$ for all $x \in X$ and all $i \in \mathbb{N}$, and if the positive term series $\sum M_i$ converges, then the sequence of partial sums, (s_n) where $s_n = \sum_{i=1}^{n} f_i(a)$, converges uniformly on X. With $M_i = (1/3)(2/3)^{i-1}$, the uniform convergence and hence the continuity of g follows (Munkres also gives a direct argument of the uniform convergence).
Next, to show $g(a) = f(a)$ for all $a \in A$. Let $s_n(x) = \sum_{i=1}^{n} g_i(x)$. Since

$$\left| f(a) - \sum_{i=1}^{n} g_i(a) \right| = |f(a) - s_n(a)| \le (2/3)^n$$

for all $a \in A$. So as $n \to \infty$, $s_n(a) \to f(a)$. That is,
$g(a) = \lim_{n\to\infty} s_n(a) = f(a)$ for all $a \in A$.

Theorem 35.1. Tietze Extension Theorem (continued 4)

Proof (continued). Finally, we show that g maps X into $[-1,1]$. This follows as above since

$$\begin{aligned} |g(x)| &= \left| \sum_{n=1}^{\infty} g_n(x) \right| = \left| \lim_{N\to\infty} \sum_{n=1}^{N} g_n(x) \right| \\ &= \lim_{N\to\infty} \left| \sum_{n=1}^{N} g_n(x) \right| \text{ since the metric on } \mathbb{R} \text{ is continuous} \\ &\le \lim_{N\to\infty} \sum_{n=1}^{N} |g_n(x)| \text{ by the Triangle Inequality} \\ &\le \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = 1. \end{aligned}$$

Therefore, part (a) of the result follows.

Theorem 35.1. Tietze Extension Theorem (continued 5)

Proof (continued).
Step 3. Now to prove part (b) of the theorem. Since the open interval $(-1,1)$ is homeomorphic to $\mathbb{R}$ (under $x \mapsto x/(x^2-1)$, say), we can replace $\mathbb{R}$ by $(-1,1)$. By the already proved part (a), we know that f can be extended to a continuous $g : X \to [-1,1]$ (into). Now we need to modify g to a function h which maps X into $(-1,1)$ (in fact the event that the image of X under g includes -1 or 1).
Given g as above, define $D \subset X$ by $D = g^{-1}(\{-1\}) \cup g^{-1}(\{a\})$. Since g is continuous, by Theorem 18.1(3), D is closed in X. Now $g(A) = f(A)$ (by Step 2) and $f(A) \subset (-1,1)$ by hypothesis, so set A is disjoint from set D. Since D is closed, by Urysohn's Lemma (Theorem 33.1) there is continuous $\varphi : X \to [0,1]$ such that $\varphi(D) = \{0\}$ and $\varphi(A) = \{1\}$. Define $h(x) = \varphi(x)g(x)$. Then h is continuous and for $a \in A$ we have
$h(a) = \varphi(a)g(a) = 1 \cdot g(a) = g(a) = f(a)$, so that h is an extension of f from Set A to all of X.

Theorem 35.1. Tietze Extension Theorem (continued 6)

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Proof (continued). For $x \in D$, $h(x) = 0 \cdot g(x) = 0$ and for $x \notin D$, $|h(x)| = |\varphi(x)g(x)| \leq |g(x)| < 1$ (since the only x for which $|g(x)| = 1$ are $x \in D$). Therefore, $h : X \to (-1, 1)$ and h is the desired extension of f from X to $(-1, 1)$ (and so f can be composed with $x/(x^2 - 1)$ to give the extension of f from X to $\mathbb{R}$). So part (b) follows. □