

## Theorem 36.1. Existence of Finite Partitions of Unity

## Introduction to Topology

## Chapter 4. Countability and Separation Axioms

## Section 36. Embeddings of Manifolds—Proofs of Theorems



0

Introduction to Topology

September 17, 2016

1 / 8

Theorem 36.1. Existence of Finite Partitions of Unity

## Theorem 36.1. Existence of Finite Partitions of Unity

(continued 1)

**Proof (continued).** Then, as above,  $A$  is closed and  $A \subset U_k$ . By Lemma

31.1.(b), there is open set  $V_k$  with  $A \subset V_k$  and  $\overline{V_k} \subset U_k$ . Then  $\{V_1, V_2, \dots, V_k, U_{k+1}, U_{k+2}, \dots, U_n\}$  covers  $X$  and  $\overline{V_i} \subset U_i$  for  $i = 1, 2, \dots, k$ . With  $k = n$ , the claim of Step 1 follows.

**Step 2.** For the given open covering  $\{U_1, U_2, \dots, U_n\}$  of  $X$ , by Step 1 there is open covering  $\{V_1, V_2, \dots, V_n\}$  of  $X$  with  $\overline{V_i} \subset U_i$  for  $i = 1, 2, \dots, n$ . Similarly, by Step 1 there is open covering

$\{W_1, W_2, \dots, W_n\}$  of  $X$  with  $\overline{W_i} \subset V_i$  for  $i = 1, 2, \dots, n$ . For each  $i = 1, 2, \dots, n$ ,  $\overline{W_i}$  and  $X \setminus V_i$  are disjoint closed sets. Since  $X$  is regular, by Urysohn's Lemma (Theorem 33.1) there is continuous  $\psi_i: X \rightarrow [0, 1]$  such that  $\psi_i(\overline{W_i}) = \{1\}$  and  $\psi_i(X \setminus V_i) = \{0\}$ . Since  $\psi_i^{-1}(\mathbb{R} \setminus \{0\}) \subset V_i$  then the closure of  $\psi_i^{-1}(\mathbb{R} \setminus \{0\})$  (i.e., the support of  $\psi_i$ ) is a subset of  $\overline{V_i}$  and so (support  $\psi_i$ )  $\subset \overline{V_i} \subset U_i$  for  $i = 1, 2, \dots, n$ .

## Theorem 36.1. Existence of Finite Partitions of Unity

## Theorem 36.1. Existence of Finite Partitions of Unity.

Let  $\{U_1, U_2, \dots, U_n\}$  be a finite open covering of the normal space  $X$ . Then there exists a partition of unity dominated by  $\{U_i\}$ .

**Proof.**

**Step 1.** First, we claim that finite open covering  $\{U_1, U_2, \dots, U_n\}$  can be modified to an open covering  $\{V_1, V_2, \dots, V_n\}$  of  $X$  with  $\overline{V_i} \subset U_i$  for  $i = 1, 2, \dots, n$ .

We inductively construct the  $V_i$ . First,  $A = X \setminus (U_2 \cup U_2 \cup \dots \cup U_n)$  is closed and (since  $\{U_i\}$  is a covering of  $X$ ) is a subset of  $U_1$ ,  $A \subset U_1$ .

Since  $X$  is normal, by Lemma 31.1(b) there is open set  $V_1$  containing  $A$  with  $\overline{V_1} \subset U_1$ . So  $\{V_1, U_2, U_3, \dots, U_n\}$  is an open cover of  $X$ .

In general, given open sets  $V_1, V_2, \dots, V_{k-1}$  such that  $\{V_1, V_2, \dots, V_{k-1}, U_k, U_{k+1}, \dots, U_n\}$  covers  $X$  and  $\overline{V_i} \subset U_i$  for  $i = 1, 2, \dots, k-1$ , let

$$A = X \setminus (V_1 \cup V_2 \cup \dots \cup V_{k-1}) \setminus (U_{k+1} \cup U_{k+2} \cup \dots \cup U_n).$$

0

Introduction to Topology

September 17, 2016

3 / 8

Theorem 36.1. Existence of Finite Partitions of Unity

## Theorem 36.1. Existence of Finite Partitions of Unity

(continued 2)

## Theorem 36.1. Existence of Finite Partitions of Unity.

Let  $\{U_1, U_2, \dots, U_n\}$  be a finite open covering of the normal space  $X$ .

Then there exists a partition of unity dominated by  $\{U_i\}$ .

**Proof (continued).** Since  $\{W_i\}$  covers  $X$ , for each  $x \in X$ , we have  $\psi_i(x) = 1$  for some  $i = 1, 2, \dots, n$  and so  $\Psi(x) = \sum_{i=1}^n \psi_i(x)$  is positive for each  $x \in X$ . Define  $\varphi_i(x) = \psi_i(x) / \Psi(x)$  (so that  $\varphi_i$  is  $\psi_i$

“normalized”). Then (support  $\varphi_i$ )  $\subset$  (support  $\psi_i$ )  $\subset U_i$  and claim (1) of the definition of “partition of unity” holds, and  $\sum_{i=1}^n \varphi_i(x) = 1$  for all  $x \in X$  and so claim (2) holds. □

0

Introduction to Topology

September 17, 2016

4 / 8

0

Introduction to Topology

September 17, 2016

5 / 8

## Theorem 36.2

**Theorem 36.2.** If  $X$  is a compact  $m$ -manifold then  $X$  can be embedded in  $\mathbb{R}^N$  for some  $N \in \mathbb{N}$ .

**Proof.** By definition, at each point  $x$  of the  $m$ -manifold there is a neighborhood of  $x$  that is homeomorphic to an open subset of  $\mathbb{R}^m$  (that is, the neighborhood can be embedded in  $\mathbb{R}^m$ ). For all  $x \in X$ , take such a neighborhood and thus form an open covering of  $X$ . Since  $X$  is compact, there is a finite subcovering, say  $\{U_1, U_2, \dots, U_n\}$ . Let the embeddings be denoted by  $g_i$ , so  $g_i : U_i \rightarrow \mathbb{R}^m$  for  $i = 1, 2, \dots, n$ . Since  $X$  is compact and Hausdorff, by Theorem 32.3,  $X$  is normal. By Existence of Finite Partitions of Unity (Theorem 36.1) there are  $\varphi_1, \varphi_2, \dots, \varphi_n$  which form a partition of unity dominated by  $\{U_1, U_2, \dots, U_n\}$ . Let  $A_i = \text{support}(\varphi_i)$ . For  $i = 1, 2, \dots, n$  define  $h_i : X \rightarrow \mathbb{R}^m$  by

$$h_i(x) = \begin{cases} \varphi_i(x)g_i(x) & \text{for } x \in U_i \\ \mathbf{0} = (0, 0, \dots, 0) & \text{for } x \in X \setminus A_i. \end{cases}$$

## Theorem 36.2 (continued 2)

**Proof (continued).** By Theorem 26.6, if we show that  $F$  is one to one then we know that  $F$  is a continuous bijection from  $X$  to its image (as a subset of  $\mathbb{R}^{(m+1)n}$ ) and, since  $X$  is compact and  $\mathbb{R}^{(m+1)n}$  is Hausdorff,  $F$  is a homeomorphism with its image (and hence is an embedding in  $\mathbb{R}^N$  with  $N = (m+1)n$ ). So suppose  $F(x) = F(y)$ . Then by the definition of  $F$ ,  $\varphi_i(x) = \varphi_i(y)$  and  $h_i(x) = h_i(y)$  for  $i = 1, 2, \dots, n$ . Now  $\varphi_{i^*}(x) > 0$  for some  $i^*$  since  $\sum_{i=1}^n \varphi_i(x) = 1$ , so  $\varphi_{i^*}(y) > 0$  and  $x, y \in \text{support}(\varphi_{i^*}) \subset U_{i^*}$ . Then

$$\begin{aligned} \varphi_{i^*}(x)g_{i^*}(x) &= h_{i^*}(x) \text{ by the definition of } h_{i^*} \text{ since } x \in U_{i^*} \\ &= h_{i^*}(y) \text{ since } h_{i^*}(x) = h_{i^*}(y) \text{ for } i = 1, 2, \dots, n \\ &= \varphi_{i^*}(y)g_{i^*}(y) \text{ by definition of } h_{i^*} \text{ since } y \in U_{i^*}. \end{aligned}$$

Since  $\varphi_{i^*}(x) = \varphi_{i^*}(y) > 0$ , we can divide to conclude  $g_{i^*}(x) = g_{i^*}(y)$ .

But  $g_{i^*} : U_{i^*} \rightarrow \mathbb{R}^m$  is an embedding and so is one to one. Therefore  $x = y$  and  $F$  is one to one. Hence,  $F$  is an embedding of  $X$  in  $\mathbb{R}^N$  where  $N = (m+1)n$ .  $\square$

## Theorem 36.2 (continued 1)

**Proof (continued).** (Notice that  $\varphi_i(x) \in \mathbb{R}$  and  $g_i(x) \in \mathbb{R}^m$ , so we interpret  $\varphi_i(x)g_i(x)$  as a scalar times a vector in vector space  $\mathbb{R}^m$ .) Since  $A_i \subset U_i$ , it is possible for  $x \in U_i \cap (X \setminus A_i)$ , but in this case  $x$  lies outside the support of  $\varphi_i$  and so  $\varphi_i(x) = 0$  and hence the “two definitions” of  $h_i$  agree (that is,  $h_i$  is well-defined). Now  $h_i$  is continuous on  $U_i$  (since  $\varphi_i$  and  $g_i$  are continuous) and  $h_i$  is continuous on  $X \setminus A_i$  (since it is constant there), so  $h_i$  is continuous on  $X = U_i \cup (X \setminus A_i)$  by the Local Formulation of Continuity (Theorem 18.2(f)).

Now define  $F : X \rightarrow (\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}) \times (\mathbb{R}^m \times \mathbb{R}^m \times \dots \times \mathbb{R}^m) = \mathbb{R}^{(m+1)n}$  ( $n$  copies of  $\mathbb{R}$  and  $n$  copies of  $\mathbb{R}^m$ ) as

$$F(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x), h_1(x), h_2(x), \dots, h_n(x)).$$

By Theorem 19.6,  $F$  is continuous.