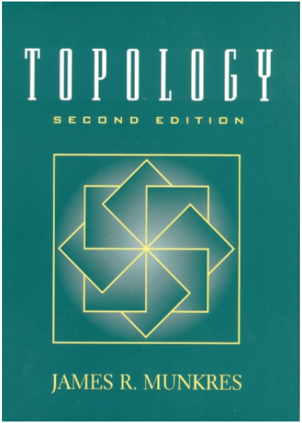
Introduction to Topology

Chapter 4. Countability and Separation Axioms

Section 36. Embeddings of Manifolds—Proofs of Theorems



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Let $\{U_1, U_2, \ldots, U_n\}$ be a finite open covering of the normal space X. Then there exists a partition of unity dominated by $\{U_i\}$.

Proof.
Step 1. First, we claim that finite open covering $\{U_1, U_2, \ldots, U_n\}$ can be modified to an open covering $\{V_1, V_2, \ldots, V_n\}$ of X with $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, n$.

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We inductively construct the V_i. First, $A = X \setminus (U_2 \cup U_2 \cup \cdots \cup U_n)$ is closed and (since $\{U_i\}$ is a covering of X) is a subset of U_1, $A \subset U_1$.

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Since X is normal, by Lemma 31.1(b) there is open set V_1 containing A with $\overline{V}_1 \subset U_1$. So $\{V_1, U_2, U_3, \ldots, U_n\}$ is an open cover of X.

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In general, given open sets $V_1, V_2, \ldots, V_{k-1}$ such that $\{V_1, V_2, \ldots, V_{k-1}, U_k, U_{k+1}, \ldots, U_n\}$ covers X and $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, k-1$, let
$A = X \setminus (V_1 \cup V_2 \cup \cdots \cup V_{k-1}) \setminus (U_{k+1} \cup U_{k+2} \cup \cdots U_n)$.

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Theorem 36.1. Existence of Finite Partitions of Unity (continued 1)

Proof (continued). Then, as above, A is closed and $A \subset U_k$. By Lemma 31.1.(b), there is open set V_k with $A \subset V_k$ and $\overline{V}_k \subset U_k$. Then $\{V_1, V_2, \ldots, V_k, U_{k+1}, U_{k+2}, \ldots, U_n\}$ covers X and $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, k$. With $k = n$, the claim of Step 1 follows.

Theorem 36.1. Existence of Finite Partitions of Unity (continued 1)

Proof (continued). Then, as above, A is closed and $A \subset U_k$. By Lemma 31.1.(b), there is open set V_k with $A \subset V_k$ and $\overline{V}_k \subset U_k$. Then $\{V_1, V_2, \ldots, V_k, U_{k+1}, U_{k+2}, \ldots, U_n\}$ covers X and $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, k$. With $k = n$, the claim of Step 1 follows.

Step 2. For the given open covering $\{U_1, U_2, \ldots, U_n\}$ of X, by Step 1 there is open covering $\{V_1, V_2, \ldots, V_n\}$ of X with $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, n$. Similarly, by Step 1 there is open covering $\{W_1, W_2, \ldots, W_n\}$ of X with $\overline{W}_i \subset V_i$ for $i = 1, 2, \ldots, n$.

Theorem 36.1. Existence of Finite Partitions of Unity (continued 1)

Proof (continued). Then, as above, A is closed and $A \subset U_k$. By Lemma 31.1.(b), there is open set V_k with $A \subset V_k$ and $\overline{V}_k \subset U_k$. Then $\{V_1, V_2, \ldots, V_k, U_{k+1}, U_{k+2}, \ldots, U_n\}$ covers X and $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, k$. With $k = n$, the claim of Step 1 follows.

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Theorem 36.1. Existence of Finite Partitions of Unity (continued 1)

Proof (continued). Then, as above, A is closed and $A \subset U_k$. By Lemma 31.1.(b), there is open set V_k with $A \subset V_k$ and $\overline{V}_k \subset U_k$. Then $\{V_1, V_2, \ldots, V_k, U_{k+1}, U_{k+2}, \ldots, U_n\}$ covers X and $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, k$. With $k = n$, the claim of Step 1 follows.

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Theorem 36.1. Existence of Finite Partitions of Unity (continued 2)

Theorem 36.1. Existence of Finite Partitions of Unity.
Let $\{U_1, U_2, \ldots, U_n\}$ be a finite open covering of the normal space X. Then there exists a partition of unity dominated by $\{U_i\}$.

Proof (continued). Since $\{W_i\}$ covers X, for each $x \in X$, we have $\psi_i(x) = 1$ for some $i = 1, 2, \ldots, n$ and so $\Psi(x) = \sum_{n=1}^{n} \psi_i(x)$ is positive for each $x \in X$. Define $\varphi_i(x) = \psi_i(x)/\Psi(x)$ (so that φ_i is ψ_i "normalized"). Then (support φ_i) = (support ψ_i) $\subset U_i$ and claim (1) of the definition of "partition of unity" holds, and $\sum_{i=1}^{n} \varphi_i(x) = 1$ for all $x \in X$ and so claim (2) holds. □

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Theorem 36.2

Theorem 36.2. If X is a compact m-manifold then X can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Proof. By definition, at each point x of the m-manifold there is a neighborhood of x that is homeomorphic to an open subset of $\mathbb{R}^m$ (that is, the neighborhood can be embedded in $\mathbb{R}^m$). For all $x \in X$, take such a neighborhood and thus form an open covering of X.

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$$h_i(x) = \begin{cases} \varphi_i(x) g_i(x) & \text{for } x \in U_i \\ \mathbf{0} = (0, 0, \ldots, 0) & \text{for } x \in X \setminus A_i. \end{cases}$$

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Proof (continued). (Notice that $\varphi_i(x) \in \mathbb{R}$ and $g_i(x) \in \mathbb{R}^m$, so we interpret $\varphi_i(x)g_i(x)$ as a scalar times a vector in vector space $\mathbb{R}^m$.) Since $A_i \subset U_i$, it is possible for $x \in U_i \cap (X \setminus A_i)$, but in this case x lies outside the support of φ_i and so $\varphi_i(x) = 0$ and hence the "two definitions" of h_i agree (that is, h_i is well-defined). Now h_i is continuous on U_i (since φ_i and g_i are continuous) and h_i is continuous on $X \setminus A_i$ (since it is constant there), so h_i is continuous on $X = U_i \cup (X \setminus A_i)$ by the Local Formulation of Continuity (Theorem 18.2(f)).

Theorem 36.2 (continued 1)

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Now define $F : X \to (\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}) \times (\mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m) = \mathbb{R}^{(m+1)n}$ (n copies of $\mathbb{R}$ and n copies of $\mathbb{R}^m$) as

$$F(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), h_1(x), h_2(x), \ldots, h_n(x)).$$

By Theorem 19.6, F is continuous.

Theorem 36.2 (continued 1)

Proof (continued). (Notice that $\varphi_i(x) \in \mathbb{R}$ and $g_i(x) \in \mathbb{R}^m$, so we interpret $\varphi_i(x)g_i(x)$ as a scalar times a vector in vector space $\mathbb{R}^m$.) Since $A_i \subset U_i$, it is possible for $x \in U_i \cap (X \setminus A_i)$, but in this case x lies outside the support of φ_i and so $\varphi_i(x) = 0$ and hence the "two definitions" of h_i agree (that is, h_i is well-defined). Now h_i is continuous on U_i (since φ_i and g_i are continuous) and h_i is continuous on $X \setminus A_i$ (since it is constant there), so h_i is continuous on $X = U_i \cup (X \setminus A_i)$ by the Local Formulation of Continuity (Theorem 18.2(f)).

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$$F(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), h_1(x), h_2(x), \ldots, h_n(x)).$$

By Theorem 19.6, F is continuous.

Theorem 36.2 (continued 2)

Proof (continued). By Theorem 26.6, if we show that F is one to one then we know that F is a continuous bijection from X to its image (as a subset of $\mathbb{R}^{(m+1)n}$) and, since X is compact and $\mathbb{R}^{(m+1)n}$ is Hausdorff, F is a homeomorphism with its image (and hence is an embedding in $\mathbb{R}^N$ with $N = (m+1)n$). So suppose $F(x) = F(y)$. Then by the definition of F, $\varphi_i(x) = \varphi_i(y)$ and $h_i(x) = h_i(y)$ for $i = 1, 2, \ldots, n$. Now $\varphi_{i^*}(x) > 0$ for some i^* since $\sum_{i=1}^{n} \varphi_i(x) = 1$, so $\varphi_{i^*}(y) > 0$ and $x, y \in \text{support}(\varphi_i) \subset U_i$. Then

$$\begin{aligned} \varphi_{i^*}(x) g_{i^*}(x) &= h_{i^*}(x) \text{ by the definition of } h_{i^*} \text{ since } x \in U_{i^*} \\ &= h_{i^*}(y) \text{ since } h_{i^*}(x) = h_{i^*}(y) \text{ for } i = 1, 2, \ldots, n \\ &= \varphi_{i^*}(y) g_{i^*}(y) \text{ by definition of } h_{i^*} \text{ since } y \in U_{i^*}. \end{aligned}$$

Theorem 36.2 (continued 2)

Proof (continued). By Theorem 26.6, if we show that F is one to one then we know that F is a continuous bijection from X to its image (as a subset of $\mathbb{R}^{(m+1)n}$) and, since X is compact and $\mathbb{R}^{(m+1)n}$ is Hausdorff, F is a homeomorphism with its image (and hence is an embedding in $\mathbb{R}^N$ with $N = (m+1)n$). So suppose $F(x) = F(y)$. Then by the definition of F, $\varphi_i(x) = \varphi_i(y)$ and $h_i(x) = h_i(y)$ for $i = 1, 2, \ldots, n$. Now $\varphi_{i^*}(x) > 0$ for some i^* since $\sum_{i=1}^n \varphi_i(x) = 1$, so $\varphi_{i^*}(y) > 0$ and $x, y \in \text{support}(\varphi_i) \subset U_i$. Then

$$\begin{aligned}
\varphi_{i^*}(x)g_{i^*}(x) &= h_{i^*}(x) \text{ by the definition of } h_{i^*} \text{ since } x \in U_{i^*} \\
&= h_{i^*}(y) \text{ since } h_{i^*}(x) = h_{i^*}(y) \text{ for } i = 1, 2, \ldots, n \\
&= \varphi_{i^*}(y)g_{i^*}(y) \text{ by definition of } h_{i^*} \text{ since } y \in U_{i^*}.
\end{aligned}$$

Since $\varphi_{i^*}(x) = \varphi_{i^*}(y) > 0$, we can divide to conclude $g_{i^*}(x) = g_{i^*}(y)$. But $g_{i^*} : U_{i^*} \to \mathbb{R}^m$ is an embedding and so is one to one. Therefore $x = y$ and F is one to one. Hence, F is an embedding of X in $\mathbb{R}^N$ where $N = (m+1)n$. □

Theorem 36.2 (continued 2)

Proof (continued). By Theorem 26.6, if we show that F is one to one then we know that F is a continuous bijection from X to its image (as a subset of $\mathbb{R}^{(m+1)n}$) and, since X is compact and $\mathbb{R}^{(m+1)n}$ is Hausdorff, F is a homeomorphism with its image (and hence is an embedding in $\mathbb{R}^N$ with $N = (m+1)n$). So suppose $F(x) = F(y)$. Then by the definition of F, $\varphi_i(x) = \varphi_i(y)$ and $h_i(x) = h_i(y)$ for $i = 1, 2, \ldots, n$. Now $\varphi_{i^*}(x) > 0$ for some i^* since $\sum_{i=1}^n \varphi_i(x) = 1$, so $\varphi_{i^*}(y) > 0$ and $x, y \in \text{support}(\varphi_i) \subset U_i$. Then

$$\begin{aligned} \varphi_{i^*}(x)g_{i^*}(x) &= h_{i^*}(x) \text{ by the definition of } h_{i^*} \text{ since } x \in U_{i^*} \\ &= h_{i^*}(y) \text{ since } h_{i^*}(x) = h_{i^*}(y) \text{ for } i = 1, 2, \ldots, n \\ &= \varphi_{i^*}(y)g_{i^*}(y) \text{ by definition of } h_{i^*} \text{ since } y \in U_{i^*}. \end{aligned}$$

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