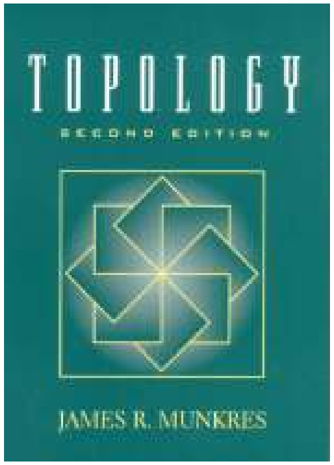
Introduction to Topology

Chapter 5. The Tychonoff Theorem
Section 37. The Tychonoff Theorem—Proofs of Theorems



Lemma 37.1

Lemma 37.1. Let X be a set. Let $\mathcal{A}$ be a set ("collection") of subsets of X having the finite intersection property. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$ and $\mathcal{D}$ has the finite intersection property, and no collection of subsets of X that properly contains $\mathcal{D}$ has this property. Such a collection $\mathcal{D}$ is said to be *maximal* with respect to the finite intersection property.

Proof. Recall Zorn's Lemma: "Let A be a set that is partially ordered. If every simply ordered subset (see page 24) of A has an upper bound in A, then A has a maximal element." In this proof, we consider sets whose elements are sets of subsets of X (so sets of sets of subset of X). Munkres calls such a set a *superset* (notice that such a superset is a subset of $\mathcal{P}(\mathcal{P}(X))$, where $\mathcal{P}(X)$ is the set of all subsets of X [the *power set* of X]) and denotes these with "black board" fonts $(\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots)$.

Lemma 37.1 (continued 1)

Proof (continued). Using a variant of the letter "C," Munkres uses the notation:

c is an element of X,
C is a subset of set X,
$\mathcal{C}$ is a collection of subsets of X (so $\mathcal{C} \subset \mathcal{P}(X)$),
$\mathbb{C}$ is a superset of set X (so $\mathbb{C} \subset \mathcal{P}(\mathcal{P}(X))$.

Let $\mathcal{A}$ be a set of subset of X having the finite intersection property. Let $\mathbb{A}$ be the superset consisting of all sets $\mathcal{B}$ of subsets of X such that $\mathcal{B} \supset \mathcal{A}$ and $\mathcal{B}$ has the finite intersection property:

$$\mathbb{A} = \{\mathcal{B} \subset \mathcal{P}(X) \mid \mathcal{B} \supset \mathcal{A}, \mathcal{B} \text{ has the finite intersection property}\}.$$

Use proper inclusion, $\subsetneq$, to give a (strict) partial order on $\mathbb{A}$. We now use Zorn's Lemma to prove that $\mathbb{A}$ has a maximal element $\mathcal{D}$.

Lemma 37.1 (continued 1)

Proof (continued). To apply Zorn's lemma, we must show that for any $\mathbb{B} \subset \mathbb{A}$ that is simply ordered, $\mathbb{B}$ has an upper bound in $\mathbb{A}$. We will show that $\mathcal{C} = \cup_{\mathcal{B} \in \mathbb{B}} \mathcal{B} \subset \mathcal{P}(X)$ is (1) in $\mathbb{A}$, and (2) is an upper bound of $\mathbb{B}$ (that is, $\mathcal{B} \subset \mathcal{C}$ for all $\mathcal{B} \in \mathbb{B}$).
To show that $\mathcal{C} \in \mathbb{A}$, we need to show that $\mathcal{C} \supset \mathcal{A}$ and that $\mathcal{C}$ has the finite intersection property. Since each $\mathcal{B} \in \mathbb{B}$ contains $\mathcal{A}$, then $\mathcal{C}$ contains $\mathcal{A}$. For the finite intersection property, let $C_1, C_2, \ldots, C_n$ be elements of $\mathcal{C}$. For each $i = 1, 2, \ldots, n$, there is $\mathcal{B}_i \in \mathbb{B}$ with $C_i \in \mathcal{B}_i$. The superset $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\} \subset \mathbb{B}$, so it is simply ordered by proper subset inclusion (since $\mathbb{B}$, by hypothesis, is simply ordered). Since the superset is finite, it has a largest element with respect to the ordering; that is, there is some k, $1 \le k \le n$, with $\mathcal{B}_i \subset \mathcal{B}_k$ for all $i = 1, 2, \ldots, n$. Then $C_i \in \mathcal{B}_k$ for all $i = 1, 2, \ldots, n$ and since $\mathcal{B}_k$ has the finite intersection property, $C_1 \cap C_2 \cap \cdots \cap C_n \neq \varnothing$. Since $C_1, C_2, \ldots, C_n$ are arbitrary elements of $\mathcal{C}$, the $\mathcal{C}$ has the finite intersection property. Therefore, $\mathcal{C} \in \mathbb{A}$.

Lemma 37.1 (continued 2)

Lemma 37.1. Let X be a set. Let $\mathcal{A}$ be a set ("collection") of subsets of X having the finite intersection property. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$ and $\mathcal{D}$ has the finite intersection property, and no collection of subsets of X that properly contains $\mathcal{D}$ has this property. Such a collection $\mathcal{D}$ is said to be *maximal* with respect to the finite intersection property.

Proof (continued). By definition of $\mathcal{C}$, $\mathcal{B} \subsetneq \mathcal{C}$ for all $\mathcal{B} \in \mathbb{B}$, so $\mathcal{C}$ is an upper bound of $\mathbb{B}$. Therefore, every simply ordered $\mathbb{B} \subset \mathbb{A}$ has any upper bound $\mathcal{C}$ in $\mathbb{A}$. So by Zorn's Lemma, there is $\mathcal{D} \subset \mathcal{P}(X)$ that has the finite intersection property such that $\mathcal{D}$ is maximal in $\mathbb{A}$. That is, $\mathcal{D}$ has the the finite intersection property and is not properly contained in another subset of $\mathcal{P}(X)$ which has the finite intersection property. □

Lemma 37.2

Lemma 37.2. Let X be a set. Let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:

(a) Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

(b) If A is a subset of X that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.

Proof. (a) Let B equal the intersection of finitely many elements of $\mathcal{D}$. Define a collection $\mathcal{E} = \mathcal{D} \cup \{B\}$. To show $\mathcal{E}$ has the finite intersection property, take finitely many elements of $\mathcal{E}$. If none of them is the set B, then their intersection is nonempty because $\mathcal{D}$ has the finite intersection property. If one of them is set B, then their intersection is of the form $D_1 \cap D_2 \cap \cdots \cap D_m \cap B$. Since B is the intersection of finitely many elements of $\mathcal{D}$, then so is this set and hence this set is nonempty. Therefore $\mathcal{E}$ has the finite intersection property and $\mathcal{D} \subset \mathcal{E}$. Since $\mathcal{D}$ is maximal with respect to the finite intersection property, then we must have $\mathcal{D} = \mathcal{E}$ and so $B \subset \mathcal{D}$ and (a) follows.

Lemma 37.2 (continued)

Lemma 37.2. Let X be a set. Let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:

(a) Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

(b) If A is a subset of X that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.

Proof (continued). (b) Let $A \subset X$ intersect every element of $\mathcal{D}$ and define $\mathcal{E} = \mathcal{D} \cup \{A\}$. Take finitely many elements of $\mathcal{E}$. If none of these elements is set A, then their intersection is nonempty by the finite intersection property of $\mathcal{D}$. If one of these elements is set A, then the intersection is of the form $D_1 \cap D_2 \cap \cdots \cap D_n \cap A$. Now $D_1 \cap D_2 \cap \cdots \cap D_n \in \mathcal{D}$ by part (a) and so the intersection is nonempty by the hypothesis that set A intersect every element of $\mathcal{D}$. Therefore $\mathcal{E}$ has the finite intersection property and as in the proof of (a) $\mathcal{E} = \mathcal{D}$ and so $A \in \mathcal{D}$. □

Theorem 37.3. The Tychonoff Theorem

Theorem 37.3. Tychonoff Theorem.
An arbitrary product of compact spaces is compact in the product topology.

Proof. Let $X = \prod_{\alpha \in J} X_\alpha$ where each X_α is compact. Let $\mathcal{A}$ be a collection of closed subsets of X having the finite intersection property. We will prove that the intersection $\cap_{A \in \mathcal{A}} A$ is nonempty and then by Theorem 26.9 it will follow that X is compact. (Munkres does not assume that the sets $A \in \mathcal{A}$ are closed and gives a more general proof, but we only need to consider a closed collection of sets to apply Theorem 26.9, so this proof assumes the sets $A \in \mathcal{A}$ are closed.)

By Lemma 37.1, choose collection $\mathcal{D}$ of subsets of X such that $\mathcal{D} \supset \mathcal{A}$ and $\mathcal{D}$ is maximal with respect to the finite intersection property. If we show that $\cap_{D \in \mathcal{D}} \overline{D} \neq \varnothing$ then (since $\mathcal{A} \subset \mathcal{D}$) it follows that $\cap_{A \in \mathcal{A}} A \neq \varnothing$ because $\cap_{D \in \mathcal{D}} \overline{D} \subset \cap_{A \in \mathcal{A}} A$.

Theorem 37.3. The Tychonoff Theorem (continued 1)

Proof (continued). Given $\alpha \in J$, consider the collection $\{\pi_\alpha(D) \mid D \in \mathcal{D}\} \subset X_\alpha$ where $\pi_\alpha : X \to X_\alpha$ is the projection map. Notice that this collection has the finite intersection property because $\mathcal{D}$ does. (Consider $\pi_\alpha(D_1) \cap \pi_\alpha(D_2) \cap \cdots \cap \pi_\alpha(D_n)$. Since $\mathcal{D}$ has the finite intersection property then there is $\mathbf{x} = (x_\alpha) \in D_1 \cap D_2 \cap \cdots \cap D_n$ and so $x_\alpha \in \pi_\alpha(D_1) \cap \pi_\alpha(D_2) \cap \cdots \cap \pi_\alpha(D_n)$.) Since X_α is compact, by Theorem 26.9, for each $\alpha \in J$ there is $x_\alpha \in X_\alpha$ such that $x_\alpha \in \cap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$. Define $\mathbf{x} = (x_\alpha) \in X$. We will show that $\mathbf{x} \in \overline{D}$ for every $D \in \mathcal{D}$ and so then $\mathbf{x} \in \cap_{A \in \mathcal{A}} A$ and the claim will follow (form Theorem 26.9).
Recall that a subbasis for the product topology includes all sets of the form $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β (see page 114). Let $\pi_\beta^{-1}(U_\beta)$ be a subbasis element containing the point $\mathbf{x} = (x_\alpha)$ of the previous paragraph. So U_β is a neighborhood of x_β in X_β. Since $x_\beta \in \overline{\pi_\beta(D)}$ for all $D \in \mathcal{D}$, then U_β intersects $\pi_\beta(D)$ in some point $\pi_\beta(\mathbf{y})$ where $\mathbf{y} \in D$. So $\mathbf{y} \in \pi_\beta^{-1}(U_\beta) \cap D$ for all $D \in \mathcal{D}$. So by Lemma 37.2(b), every subbasis element containing $\mathbf{x}$ belongs to $\mathcal{D}$.

Theorem 37.3. The Tychonoff Theorem (continued 2)

Theorem 37.3. Tychonoff Theorem.
An arbitrary product of compact spaces is compact in the product topology.

Proof (continued). Now every basis element of the product topology is of the form

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

for some $\beta_1, \beta_2, \ldots, \beta_n$ and some $U_{\beta_i} \subset X_{\beta_i}$ for $i = 1, 2, \ldots, n$ (see page 115). Therefore by Lemma 37.2(a), every basis element containing $\mathbf{x}$ belongs to $\mathcal{D}$. Since $\mathcal{D}$ has the finite intersection property, every basis element containing $\mathbf{x}$ intersects every element of $\mathcal{D}$ (applying the finite intersection property to the two sets, a basis element and an element of $\mathcal{D}$). So $\mathbf{x} \in \overline{D}$ for every $D \in \mathcal{D}$. Therefore, $\cap_{D \in \mathcal{D}} D \neq \varnothing$. As discussed above, by Theorem 26.9, $C = \prod_{\alpha \in J} X_\alpha$ is compact. □