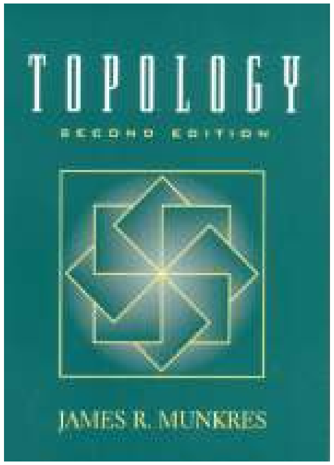
Introduction to Topology

Chapter 6. Metrization Theorems and Paracompactness

Section 39. Local Finiteness—Proofs of Theorems



Lemma 39.1

Lemma 39.1. Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

(a) Any subcollection of $\mathcal{A}$ is locally finite.

(b) The collection $\mathcal{B} = \{\overline{A}\}_{A\in\mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.

(c) $\overline{\cup_{A\in\mathcal{A}} A} = \cup_{A\in\mathcal{A}} \overline{A}$.

Proof. (a) This follows trivially from the definition.

(b) First, note that any open set U that intersects set $\overline{A}$ must also intersect A (since $\overline{A} = A \cup A'$ where A' is the set of limit points of A, by Theorem 17.6). So if U is a neighborhood of $x \in X$ that only intersects finitely many $A \in \mathcal{A}$, say $A_1, A_2, \ldots, A_n$, then U also only intersects $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n \in \mathcal{B}$ (see Theorem 17.5(a); it could be that $\overline{A}_i = \overline{A}_j$ and U could actually intersect fewer elements of $\mathcal{B}$ than of $\mathcal{A}$).

Lemma 39.1 (continued)

Lemma 39.1. Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

(c) $\overline{\cup_{A\in\mathcal{A}} A} = \cup_{A\in\mathcal{A}} \overline{A}$.

Proof (continued). (c) Denote $Y = \cup_{A\in\mathcal{A}} A$. Now each $A \in \mathcal{A}$ is a subset of Y so $\overline{A} \subset \overline{Y}$ (apply Theorem 17.5(a), say). Now let $x \in \overline{Y}$ and let U be a neighborhood of x. Then, since $\mathcal{A}$ is locally finite in X, U intersects only finitely many elements of $\mathcal{A}$, say $A_1, A_2, \ldots, A_k$. ASSUME $x \not\in \overline{A}_1, x \not\in \overline{A}_2, \ldots, x \not\in \overline{A}_k$. Then set $\setminus(\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_k)$ is a neighborhood of x that intersects no element of $\mathcal{A}$. But then U is a neighborhood of x that does not intersect $Y = \cup_{A\in\mathcal{A}} A$, a CONTRADICTION to the fact that $x \in \overline{Y}$ (see Theorem 17.5(a)). So it must be that $x \in \overline{A}_i$ for some i and hence $x \in \cup_{A\in\mathcal{A}} \overline{A}$. Therefore $\overline{Y} \subset \cup_{A\in\mathcal{A}} \overline{A}$ and so $\overline{Y} = \cup_{A\in\mathcal{A}} \overline{A}$, as claimed. □

Lemma 39.2

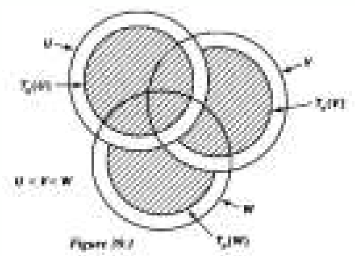
Lemma 39.2. Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering $\mathcal{E}$ of X refining $\mathcal{A}$ that is countable locally finite.

Proof. We will use the Well-Ordering Theorem: "If A is a set, there exists an order relation on A that is a well-ordering." Recall that this is equivalent to the Axiom of Choice. Let $<$ be a well-ordering for set $\mathcal{A}$.

Since X is metrizable, there is a metric d on X. Let $n \in \mathbb{N}$. Given $U \in \mathcal{A}$, define $S_n(U)$ as the subset of U obtained by "shrinking" U a distance of $1/n$: $S_n(U) = \{x \mid B(x, 1/n) \subset U\}$. For each $U \in \mathcal{A}$, define $T_n(U) = S_n(U) \setminus \cup_{V\in\mathcal{A}, V<U} V$. The resulting $T_n(U)$ are then disjoint ($T_n(U) \subset S_n(U) \subset U$, so for any $U_1, U_2 \in \mathcal{A}$, we have, say $U_1 < U_2$ and so $T_n(U_1) \cap T_n(U_2) = \varnothing$).

Lemma 39.2 (continued 1)

Proof (continued). Let $V, W \in \mathcal{A}$ with $V \neq W$. If $x \in T_n(V)$ and $y \in T_n(W)$ then we claim $d(x,y) \geq 1/n$ (see Figure 39.1 in which $U < V < W$).



To justify this, say $V < W$. Since $x \in T_n(V) \subset S_n(V)$, then the $1/n$-neighborhood of x lies in V (by the definition of $S_n(V)$). Since $V < W$ and $y \in T_n(V)$ then $y \notin V$ (by the definition of $T_n(W)$), and so y is not in the $1/n$-neighborhood of x.

Lemma 39.2 (continued 2)

Proof (continued). Now for each $U \in \mathcal{A}$, define

$$E_n(U) = \{B(x, 1/(3n)) \mid x \in T_n(U)\}$$

where $B(x, 1/(3n)) = \{y \in X \mid d(x,y) < 1/(3n >\}$. That is, $E_n(U)$ is an "expansion" of $T_n(U)$ by an amount of $1/(3n)$. Notice that $E_n(U) \subset U$ and since $E_n(U)$ is a union of "open balls" then $E_n(U)$ itself is open. Let $V, W \in \mathcal{A}$ with $V \neq W$. If $x \in E_n(V)$ and $y \in E_n(W)$ then we claim $d(x,y) \geq 1/(3n)$ (see Figure 39.2 in which $U < V < W$).

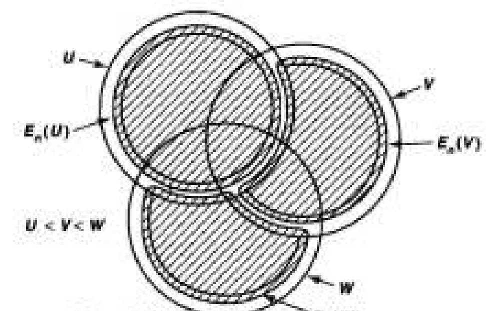


Figure 39.2

Lemma 39.2 (continued 3)

Proof (continued). By the construction of $E_n(V)$ and $E_n(W)$, there are $x' \in T_n(V)$ and $y' \in T_n(W)$ such that $d(x,x') \leq 1/(3n)$ and $d(y,y') \leq 1/(3n)$. As observed above, $d(x',y') \geq 1/n$ for such x and y. So

$$\frac{1}{n} \leq d(x',y') \leq d(x',x) + d(x,y) + d(y,y') \text{ by the Triangle Inequality}$$
$$\leq \frac{1}{3n} + d(x,y) + \frac{1}{3n},$$

or $d(x,y) \geq 1/(3n)$.
Now define $\mathcal{E}_n = \{E_n(U) \mid U \in \mathcal{A}\}$. We claim that $\mathcal{E}_n$ is a locally finite collection of open sets that refines $\mathcal{S}$. First, by construction, each $E_n(U)$ is open and refines $\mathcal{A}$ since $E_n(U) \subset U$ for all $U \in \mathcal{A}$. For any $x \in X$, the $1/(6n)$-neighborhood of x intersects at most one element of $\mathcal{E}_n$ (since the elements of $\mathcal{E}_n$ are a distance of at least $1/(3n)$ apart). So $\mathcal{E}_n$ is locally finite.

Lemma 39.2 (continued 4)

Lemma 39.2. Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering $\mathcal{E}$ of X refining $\mathcal{A}$ that is countable locally finite.

Proof (continued). Now $\mathcal{E}_n$ may not cover X for any given $n \in \mathbb{N}$ (see Figure 39.2), so consider $\mathcal{E} = \cup_{n=1}^{\infty} \mathcal{E}_n$. Let $x \in X$. We hypothesized that $\mathcal{A}$ was a covering of X, so use the well-ordering on $\mathcal{A}$ to choose U as the "first" (that is, $<$-least) element of $\mathcal{A}$ that contains x. Since U is open (by hypothesis), there is some $n \in \mathbb{N}$ such that $B(x, 1/n) \subset U$ (since the topology on X is hypothesized to be the metric topology under metric d). Then by the definition of $S_n(U)$, $x \in S_n(U)$. Since U is the "first" element of $\mathcal{A}$ that contains x, then by the definition of $T_n(U)$ we have $x \in T_n(U)$. Since $T_n(U) \subset E_n(U)$, then $x \in E_n(U)$. Therefore, $\mathcal{E}$ is a covering of X. Since each $\mathcal{E}_n$ is a refinement of $\mathcal{A}$ then $\mathcal{E}$ is a refinement of $\mathcal{A}$ and since each $\mathcal{E}_n$ is locally finite, then $\mathcal{E}$ is countably locally finite, as claimed. □