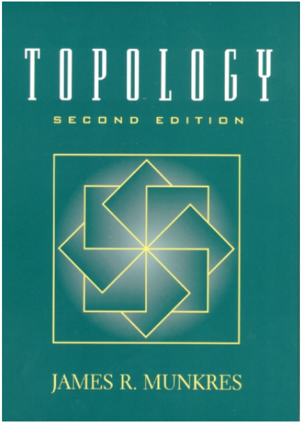
Introduction to Topology

Chapter 6. Metrization Theorems and Paracompactness

Section 39. Local Finiteness—Proofs of Theorems



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Lemma 39.1. Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

(a) Any subcollection of $\mathcal{A}$ is locally finite.

(b) The collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.

(c) $\overline{\cup_{A \in \mathcal{A}} A} = \cup_{A \in \mathcal{A}} \overline{A}$.

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Lemma 39.1 (continued)

Lemma 39.1. Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

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Proof (continued). (c) Denote $Y = \cup_{A\in\mathcal{A}} A$. Now each $A \in \mathcal{A}$ is a subset of Y so $\overline{A} \subset \overline{Y}$ (apply Theorem 17.5(a), say). Now let $x \in \overline{Y}$ and let U be a neighborhood of x. Then, since $\mathcal{A}$ is locally finite in X, U intersects only finitely many elements of $\mathcal{A}$, say $A_1, A_2, \ldots, A_k$. ASSUME $x \notin \overline{A}_1, x \notin \overline{A}_2, \ldots, x \notin \overline{A}_k$. Then set $\setminus(\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_k)$ is a neighborhood of x that intersects no element of $\mathcal{A}$. But then U is a neighborhood of x that does not intersect $Y = \cup_{A\in\mathcal{A}} A$, a CONTRADICTION to the fact that $x \in \overline{Y}$ (see Theorem 17.5(a)).

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Lemma 39.2. Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering $\mathcal{E}$ of X refining $\mathcal{A}$ that is countable locally finite.

Proof. We will use the Well-Ordering Theorem: "If A is a set, there exists an order relation on A that is a well-ordering." Recall that this is equivalent to the Axiom of Choice. Let $<$ be a well-ordering for set $\mathcal{A}$.

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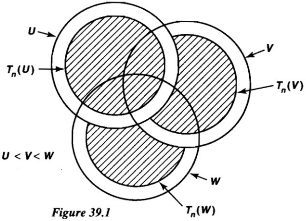
Proof (continued). Let $V, W \in \mathcal{A}$ with $V \neq W$. If $x \in T_n(V)$ and $y \in T_n(W)$ then we claim $d(x, y) \geq 1/n$ (see Figure 39.1 in which $U < V < W$).



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To justify this, say $V < W$. Since $x \in T_n(V) \subset S_n(V)$, then the $1/n$-neighborhood of x lies in V (by the definition of $S_n(V)$). Since$V < W$ and $y \in T_n(V)$ then $y \notin V$ (by the definition of $T_n(W)$), and so y is not in the $1/n$-neighborhood of x.

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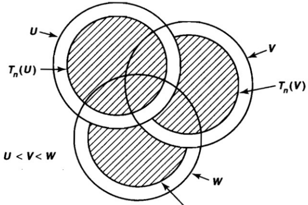

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Proof (continued). Now for each $U \in \mathcal{A}$, define

$$E_n(U) = \{B(x, 1/(3n)) \mid x \in T_n(U)\}$$

where $B(x, 1/(3n)) = \{y \in X \mid d(x, y) < 1/(3n >\}$. That is, $E_n(U)$ is an "expansion" of $T_n(U)$ by an amount of $1/(3n)$. Notice that $E_n(U) \subset U$ and since $E_n(U)$ is a union of "open balls" then $E_n(U)$ itself is open.

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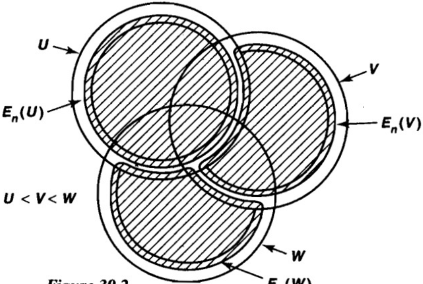

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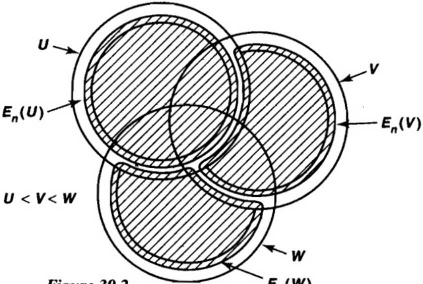

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