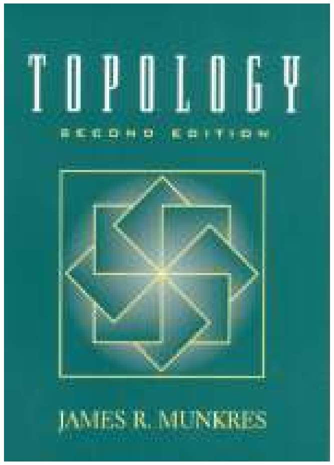
Introduction to Topology

Chapter 6. Metrization Theorems and Paracompactness

Section 40. The Nagata-Smirnov Metrization Theorem—Proofs of Theorems



Lemma 40.1

Lemma 40.1. Let X be a regular space with a basis $\mathcal{B}$ that is countably localy finite. Then X is normal, and every closed set in X is a G_δ set in X.

Proof. We follow Munkres' three-step proof.
Step 1. Let W be an open set in X. We claim there is a countable collection $\{U_n\}_{n=1}^\infty$ of open sets such that $W = \cup_{n=1}^\infty U_n = \cup_{n=1}^\infty \overline{U}_n$. Since the basis $\mathcal{B}$ for X is countably locally finite then, by definition, we can write $\mathcal{B} = \cup_{n=1}^\infty \mathcal{B}_n$ where each collection $\mathcal{B}_n$ is locally finite. Let $\mathcal{C}_n$ be the set of basis elements B such that $B \in \mathcal{B}_n$ and $\overline{B} \subset W$. Since $\mathcal{B}_n$ is locally finite (that is, every $x \in X$ has a neighborhood that intersects a finite number of elements of $\mathcal{B}_n$) then $\mathcal{C}_n \subset \mathcal{B}_n$ is locally finite. Define $U_n = \cup_{B\in\mathcal{C}_n} B$. Then U_n is open and by Lemma 39.1(c) (since U_n is locally finite) $\overline{U}_n = \cup_{B\in\mathcal{C}_n} \overline{B}$. Since each $\overline{B} \subset W$ (by the construction of $\mathcal{C}_n$) then $\overline{U}_n \subset W$, so that $\cup_{n=1}^\infty U_n \subset \cup_{n=1}^\infty \overline{U}_n \subset W$. Now for a given $x \in W$, since X is regular, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $\overline{B} \subset W$ (applying the regularity to point x and closed set $X \setminus W$).

Lemma 40.1 (continued 1)

Proof (continued). Now $B \in \mathcal{B}_n$ for some $n \in \mathbb{N}$ and by the definition of $\mathcal{C}_n$, we have $B \in \mathcal{C}_n$. Therefore $x \in U_n = \cup_{B\in\mathcal{C}_n} B$. Since x is an arbitrary element of W then $W \subset \cup_{n=1}^\infty U_n$ and since $\cup_{n=1}^\infty U_n \subset \cup_{n=1}^\infty \overline{U}_n \subset W$ then we have $W = \cup_{n=1}^\infty U_n = \cup_{n=1}^\infty \overline{U}_n$, as claimed.

Step 2. We now establish the G_δ claim. Let C be a closed set in X and let $W = X \setminus C$. Then W is open and so, by Step 1, there are open sets U_n such that $W = \cup_{n=1}^\infty \overline{U}_n$. Then

$$\begin{aligned} C &= X \setminus W = X \setminus \cup_{n=1}^\infty \overline{U}_n = X \cap \left(\cup_{n=1}^\infty \overline{U} + n\right)^c \\ &= X \cap \left(\cap_{n=1}^\infty \overline{U}_n^c\right) \text{ by De Morgan's Law} \\ &= \cap_{n=1}^\infty \overline{U}_n^c = \cup_{n=1}^\infty (X \setminus U_n). \end{aligned}$$

Since U_n is open then $X \setminus U_n$ is closed and so C is a countable intersection of closed sets. That is, C is a G_δ set.

Lemma 40.1 (continued 2)

Proof (continued).
Step 3. We now show that X is normal. Let C and D be disjoint closed sets in X. Then $X \setminus D$ is open and by Step 1 there is a countable collection $\{U_n\}_{n=1}^\infty$ of open sets such that $\cup_{n=1}^\infty U_n = \cup_{n=1}^\infty \overline{U}_n = X \setminus D$. Then $\{U_n\}_{n=1}^\infty$ covers set C (since $C \subset X \setminus D$) and each $\overline{U}_n$ is disjoint from set D. Similarly, there is a countable open covering $\{V_n\}_{n=1}^\infty$ of D with each $\overline{V}_n$ disjoint from set C. (We now repeat part of the proof of Theorem 32.1 in which we were given a countable basis and showed that this implies regularity.) Define

$$U_n' = U_n \setminus \cup_{i=1}^n V_i \text{ and } V_n' = V_n \setminus \cup_{i=1}^n \overline{U}_i.$$

Then, as shown in the proof of Theorem 32.1 (see Section 32 or pages 200–201) we have that the sets

$$U' = \cup_{n=1}^\infty U_n' \text{ and } V' = \cup_{n=1}^\infty V_n'$$

are disjoint open sets with $C \subset U'$ and $D \subset V'$. □

Lemma 40.2

Lemma 40.2. Let X be normal. Let A be a closed G_δ set. Then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$.

Proof. This was given in Section 33 as Exercise 33.4. We prove it now. Since A is a G_δ set, let $A - \cap_{n=1}^\infty U_n$ where each U_n is open. Since A is closed by hypothesis, $X \setminus U_n$ is closed, and sets A and $X \setminus U_n$ are disjoint, then by Urysohn's Lemma (Theorem 33.1) there is a continuous function $f_n : X \to [0,1]$ such that $f_n(x) = 0$ for $x \in A$ and $f_n(x) = 1$ for $x \in X \setminus U_n$. Define $f(x) = \sum_{n=1}^\infty f_n(x)/2^n$. Then the series converges uniformly on X (compare it to the geometric series $\sum_{n=1}^\infty 1/2^n$) and so f is continuous by the Uniform Limit Theorem (Theorem 21.6). Also, $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$, so f is the desired function. □

Theorem 40.3

Theorem 40.3. The Nagata-Smirnov Metrization Theorem.
A topological space X is metrizable if and only if X is regular and has a basis that is countably locally finite.

Proof. First, assume$X$ is regular with a countably locally finite basis $\mathcal{B}$. Then X is normal and every closed set in X is a G_δ set by Lemma 40.1. We shall show that X is metrizable by embedding X in the metric space $(\mathbb{R}^J, \overline{\rho})$ for some J, where $\overline{\rho}$ is the uniform metric (see Section 20 and page 124).

Let $\mathcal{B} = \cup_{n=1}^\infty \mathcal{B}_n$ where each collection $\mathcal{B}_n$ is locally finite. By Lemma 40.2, for each $n \in \mathbb{N}$ and each basis element $B \in \mathcal{B}_n$ there is a continuous function $f_{n,B} : X \to [0, 1/n]$ such that $f_{n,B}(x) > 0$ for $x \in B$ and $f_{n,B}(x) = 0$ for $x \notin B$ (where $X \setminus B$ is closed and so G_δ; notice that the continuous function of Lemma 40.2 must be scaled by a factor of $1/n$). Now for any given $x_0 \in X$ and neighborhood U of x_0, there is a basis element such that $x_0 \in B \subset U$ (by Lemma 13.1, say).

Theorem 40.3 (continued 1)

Proof (continued). Then $B \in \mathcal{B}_n$ for some $n \in \mathbb{N}$, and so there is $f_{n,B}$ with $f_{n,B}(x_0) > 0$ and $f_{n,B}(x) = 0$ for $x \notin U$. That is, $\{f_{n,B}\}$ separates points from closed sets (see Section 34).

Let J be the subset of $\mathbb{N} \times \mathcal{B}$ consisting of all pairs (n, B) such that B is an element of $\mathcal{B}_n$. Define $F : X \to [0,1]^J$ as $f(x) = (f_{n,B}(x))_{(x,B)\in J}$. So by Theorem 34.2, F is an embedding of X in $[0,1]^J$ (where $[0,1]^J$ has the product topology).

Now we give $[0,1]^J$ the topology induced by the uniform metric and show that F is an embedding relative to this topology as well. The uniform topology is finer than the product topology by Theorem 20.4. Since F is an embedding of X into $[0,1]^J$ under the product topology, then F maps each open set in X to an open set of $[0,1]^J$ under the product topology, then F maps every open set in X to an open set in $[0,1]^J$ under the uniform topology. Also, since F is an embedding then F is injective (one to one).

Theorem 40.3 (continued 2)

Proof (continued). So F is an embedding of X into $[0,1]^J$ under the uniform topology on $[0,1]^J$ (that is, inverse images of open sets under F are open in X; since the uniform topology is finer then it has "more" open sets than the product topology on $[0,1]^J$). We show this next.

Notice that on $[0,1]^J$ as a subspace of $\mathbb{R}^J$, the uniform metric is $\rho((x_\alpha), (y_\alpha)) = \sup\{|x_\alpha - y_\alpha| \mid \alpha \in J\}$. To prove continuity, let $x_0 \in X$ and let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be given. Since $\mathcal{B}_n$ is locally finite, there is a neighborhood U_n of x_0 such that U_n intersects only finitely many elements of the collection $\mathcal{B}_n$. Now as basis element B ranges over $\mathcal{B}_n$, $f_{n,B}(x) = 0$ for $x \notin B$ so that for all but finitely many such B we have $f_{n,B}(x) = 0$ for all $x \in U_n$ (since $U_n \cap B = \varnothing$ for all but finitely many B). Now for each of the remaining finite number of $f_{n,B}$ there is a neighborhood of x_0 such that on this neighborhood $f_{n,B}$ varies from $f_{n,B}(x_0)$ by less than $\varepsilon/2$. Let V_n be the intersection of these finite number of neighborhoods of x_0.

Theorem 40.3 (continued 3)

Proof (continued). Then V_n is open and contains x_0 on which ALL $f_{n,B}$ (for the given $n \in \mathbb{N}$ we are currently considering) vary from the value $f_{n,B}(x_0)$ by less than $\varepsilon/2$ (the $f_{n,B}$ other than the finitely many are constant and so don't vary at all on V_n).

Choose a neighborhood of x_0 for each $n \in \mathbb{N}$ satisfying the conditions of the previous paragraph. Choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$ and define $W = V_1 \cap V_2 \cap \cdots \cap V_N$. We now show that for each $x \in W$, $\overline{\rho}(F(x), F(x_0)) < \varepsilon$. Let $x \in W$. If $n \leq N$ then $|f_{n,B}(x) - f_{n,B}(x_0)| \leq \varepsilon/2$ because $f_{n,B}$ is either a constant of 0 or varies by at most $\varepsilon/2$ on W. If $n > N$ then $|f_{n,B}(x) - f_{n,B}(x_0)| \leq 1/n > 1/N < \varepsilon/2$ because $f_{n,B}$ maps X into $[0, 1/n]$. Therefore $\overline{\rho}(F(x), F(x_0)) \leq \varepsilon/2 < \varepsilon$ on W. Since x_0 and $\varepsilon > 0$ are arbitrary, then F is continuous on X. Therefore F is an embedding of X into $[0,1]^J$ where $[0,1]^J$ has the topology induced by metric $\overline{\rho}$. Therefore X is homeomorphic to a metric space (a subspace of $(\mathbb{R}, \overline{\rho})$) and so X is metrizable.

Theorem 40.3. The Nagata-Smirnov Metrization Theorem

Theorem 40.3 (continued 4)

Proof (continued). Now suppose X is metrizable. Then X is normal by Theorem 33.2 and therefore is regular (since every normal space is regular). Now to show X has a basis that is countably finite.
Let d be a metric on X. Given $m > 0$, let $\mathcal{A}_m$ be the covering of X by all open balls of radius $1/m$: $\mathcal{A}_m = \{B(x, 1/m) \mid x \in X\}$. By Lemma 39.2, there is an open covering of $\mathcal{B}_m$ of X refining $\mathcal{A}_m$ such that $\mathcal{B}_m$ is countably locally finite. Then each element of $\mathcal{B}_m$ has diameter of at most $2/m$. Let $\mathcal{B} = \cup_{m=1}^{\infty} \mathcal{B}_m$. Since each $\mathcal{B}_m$ is countably locally finite and a countable union of countable sets is countable (by Theorem 7.5) then $\mathcal{B} = \cup_{m=1}^{\infty} \mathcal{B}_m$ is also countably locally finite. Now to show that $\mathcal{B}$ is a basis of X.

Theorem 40.3. The Nagata-Smirnov Metrization Theorem

Theorem 40.3 (continued 5)

Theorem 40.3. The Nagata-Smirnov Metrization Theorem.
A topological space X is metrizable if and only if X is regular and has a basis that is countably locally finite.

Proof (continued). Given $x \in X$ and $\varepsilon > 0$, there is some $m \in \mathbb{N}$ with $1/m < \varepsilon/2$, there is some open covering of X (by definition of $\mathcal{B}_m$), there is some $B \in \mathcal{B}_m$ where $x \in B$. Since $x \in B$ and $2/m < \varepsilon$, then $B \subset B(x, \varepsilon)$ (where $B(x, \varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$). Since $x \in X$ and $\varepsilon > 0$ are arbitrary, then $\mathcal{B}$ is a basis for the metric topology induced by metric d (by the definition of metric topology and Lemma 13.2). Therefore X is regular and $\mathcal{B}$ is a countably locally finite basis, as claimed. □