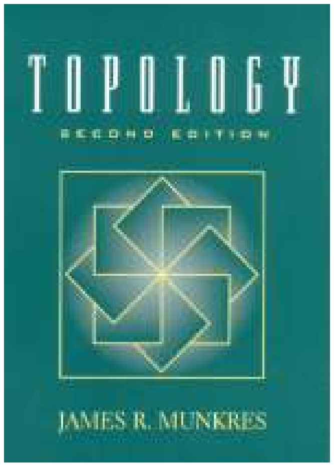
Introduction to Topology

Chapter 6. Metrization Theorems and Paracompactness

Section 41. Paracompactness—Proofs of Theorems



Theorem 41.1

Theorem 41.1. Every paracompact Hausdorff space X is normal.

Proof. We follow Munkres proof, first showing regularity and then repeating the argument to show normality.

Let $a \in X$ and let B be a closed set in X not containing a. Since X is Hausdorff, for each $b \in B$ there are open sets U_a and U_b with $a \in U_a$, $b \in U_b$, and $U_a \cap U_b = \varnothing$. Then $a \notin \overline{U_b}$ ($\overline{U_b}$ consists of the points in U_b and the limit points of U_b by Theorem 17.6; since U_a is disjoint from U_b then it cannot contain any limit points of U_b). Cover X by the collection of open sets $\mathcal{A} = \{U_b \mid b \in B\} \cup \{X \setminus B\}$. Since X is paracompact, there is a countably locally finite open cover $\mathcal{C}$ of X that refines $\mathcal{A}$. Form the subcollection $\mathcal{D}$ of $\mathcal{C}$ consisting of every element of $\mathcal{C}$ that intersects B. Then $\mathcal{D}$ covers B (since $\mathcal{C}$ covers X). Furthermore, if $D \in \mathcal{D}$, then $\overline{D}$ does not contain a since D intersects B and so it lies in some U_b (since $\mathcal{D} \subset \mathcal{C}$ and $\mathcal{C}$ is a refinement of $\mathcal{A}$) where $a \notin \overline{U_b}$ and $\overline{D} \subset \overline{U_b}$.

Theorem 41.1 (continued 1)

Proof (continued). Let $V = \cup_{D \in \mathcal{D}} D$; then V is open in X and $B \subset V$ (since $\mathcal{D}$ covers B). Since $\mathcal{C}$ is a locally finite covering of X then $\mathcal{D}$ is a locally finite covering of B. Therefore, by Lemma 39.1(c), $\overline{V} = \cup_{D \in \mathcal{D}} \overline{D}$. Since $a \notin \overline{U_b}$ and $\overline{D} \subset \overline{U_b}$, then $a \notin \overline{V}$. Since $a \notin \overline{V}$, then a is neither in V nor is a a limit point of V (see Theorem 17.6), so there is some open set U containing a such that $U \cap V = \varnothing$. So U and V are open, $U \cap V = \varnothing$, closed set $B \subset V$ and $a \in U$. That is, X is a regular topological space.

Now for normality, let A and B be closed sets in X. Since X is regular by the previous argument, for each $b \in B$ there are open sets U_b and $U_{b,A}$ with $b \in U_b$, $A \subset U_{b,A}$, and $U_b \cap U_{b,A} = \varnothing$. Then $A \cap \overline{U_b} = \varnothing$ ($U_{b,A}$ consists of the points in $U_{b,A}$ and the limit points of $U_{b,A}$ by Theorem 17.6; since $U_{b,A}$ is disjoint from U_b then it cannot contain any limit points of U_b). Cover X by the collection of open sets $\mathcal{A} = \{U_b \mid b \in B\} \cup \{X \setminus B\}$. Since X is paracompact, there is a countably locally finite covering $\mathcal{C}$ of X that refines $\mathcal{A}$.

Theorem 41.1 (continued 2)

Theorem 41.1. Every paracompact Hausdorff space X is normal.

Proof (continued). Form the subcollection $\mathcal{D}$ of $\mathcal{C}$ consisting of every element of $\mathcal{C}$ that intersects B. Then $\mathcal{D}$ covers B (since $\mathcal{C}$ covers X). Furthermore, if $D \in \mathcal{D}$ then $\overline{D} \cap A = \varnothing$ since D intersects B and so it lies in some U_b (since $\mathcal{D} \subset \mathcal{C}$ and $\mathcal{C}$ is a refinement of $\mathcal{A}$) where $A \cap \overline{U_b} = \varnothing$ and $\overline{D} \subset \overline{U_b}$. Let $V = \cup_{D \in \mathcal{D}} D$; then V is open in X and $B \subset V$ (since $\mathcal{D}$ covers B). Since $\mathcal{C}$ is a locally finite covering of X then $\mathcal{D}$ is a locally finite covering of B. Therefore, by Lemma 39.1(c), $\overline{V} = \cup_{D \in \mathcal{D}} \overline{D}$. Since $A \cap \overline{U_b} = \varnothing$ and $\overline{D} \subset \overline{U_b}$ then $A \cap \overline{D} = \varnothing$ for all $D \in \mathcal{D}$ and hence $A \cap \overline{D} = \varnothing$. Since $A \cap \overline{V} = \varnothing$, then for all $a \in A$, a is neither in V nor a limit point of V (see Theorem 17.6), so there is an open set U_a with $a \in U_a$ and $U_a \cap V = \varnothing$. Define $U = \cup_{a \in A} U_a$. Then U and V are open, $U \cap V = \varnothing$, $A \subset U$ and $B \subset V$. That is, X is a normal topological space. □

Theorem 41.2

Theorem 41.2. Every closed subspace of a paracompact space is paracompact.

Proof. Let Y be a closed subspace of the paracompact space X. Let $\mathcal{A}$ be a covering of Y by sets open in Y. For each $A \in \mathcal{A}$, choose an open set A' of X such that $A' \cap Y = A$ (which can be done by the definition of the subspace topology). Cover X by the sets A' (which are open in X), along with the open (in X) set $X \setminus Y$ (this is where Y is closed is used). Since X is paracompact, there is a locally finite open refinement $\mathcal{B}$ of the covering of X by the A''s that cover X. The collection $\mathcal{C} = \{B \cap Y \mid B \in \mathcal{B}\}$ is then an open refinement of $\mathcal{A}$ covering Y. Since $\mathcal{B}$ is locally finite then (by definition) each $x \in X$ has a neighborhood intersecting only finitely many $B \in \mathcal{B}$. Therefore, each $y \in Y$ has a neighborhood (in the subspace topology) which intersects only finitely many $B \cap Y \in \mathcal{C}$. That is, $\mathcal{C}$ is locally finite. Therefore, Y is paracompact. □

Lemma 41.3

Lemma 41.3. Let X be a regular topological space. The following conditions on X are equivalent. Every open covering of X has a refinement that is:

(1) an open covering of X and countably locally finite,
(2) a covering of X and locally finite,
(3) a closed covering of X and locally finite, and
(4) an open covering of X and locally finite (that is, X is paracompact).

Proof. (4)⇒(1). Since an open covering of X is countably locally finite (by definition) if it can be written as a countable union of collections of sets each of which is locally finite, then (4)⇒(1).

Theorem 41.3 (continued 1)

Proof (continued). (1)⇒(2). Let $\mathcal{A}$ be an open covering of X and let $\mathcal{B}$ be an open refinement of $\mathcal{A}$ that covers X and is countably locally finite (which exists by (1)). Let $\mathcal{B} = \cup_{n=1}^{\infty} \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite (but notice that the $\mathcal{B}_n$'s may not cover X). For $i \in \mathbb{N}$, let $V_i = \cup_{U \in \mathcal{B}_i} U$. For each $n \in \mathbb{N}$ and each $U \in \mathcal{B}_n$, define $S_n(U) = U \setminus \cup_{i<n} V_i$. Let $\mathcal{C}_n = \{S_n(U) \mid U \in \mathcal{B}_n\}$. Then $\mathcal{C}_n$ is a refinement of $\mathcal{B}_n$ since $S_n(U) \subset U$ for each $U \in \mathcal{B}_n$ (but $S_n(U)$ may not be open [nor closed]). Let $\mathcal{C} = \sup_{n=1}^{\infty} \mathcal{C}_n$. We claim that $\mathcal{C}$ is the required locally finite refinement of $\mathcal{A}$ covering X. Since each $\mathcal{C}_n$ is a refinement of each $\mathcal{B}_n$, then $\mathcal{C}$ is a refinement of $\mathcal{B}$ and hence of $\mathcal{A}$.

Let $x \in X$. Let N be the smallest index such that $x \in \mathcal{B}_N$ (since $\mathcal{B}$ is a covering of X, such N exists). Let $U_x \in \mathcal{B}_N$ contain x. Since $x \notin \mathcal{B}_i$ for $i < N$, then $x \in S_N(U_x) \in \mathcal{C}_N \subset \mathcal{C}$. So $\mathcal{C}$ is a covering of X.

Theorem 41.3 (continued 2)

Proof (continued). Next (to show that $\mathcal{C}$ is locally finite) since each collection $\mathcal{B}_n$ is locally finite, then for each index volume $n = 1, 2, \ldots, N$ there is a neighborhood W_n of x that intersects only finitely many elements of $\mathcal{B}_n$. Now for a given $V \in \mathcal{B}_n$, if W_n intersects $S_n(V) \in \mathcal{C}_n$ then W_n must intersect $V \in \mathcal{B}_n$ since $S_n(V) \subset V$, or by the contrapositive, if W_n does not intersect $V \in \mathcal{B}_n$ then W_n does not intersect $S_n(V) \in \mathcal{C}_n$. Since W_n intersects only finitely many elements of $\mathcal{B}_n$ then W_n intersects only finitely many elements of $\mathcal{C}_n$. Since $U_x \in \mathcal{B}_M$ (the U_x containing x introduced in the previous paragraph), then U_x intersects no element of $\mathcal{C}_n$ for $n > N$ (since $\mathcal{C}_n = \{S_n(U) \mid U \in \mathcal{B}_n\}$ and $S_n(U) = U \setminus \cup_{i<n} V_i$). So the open set $W_1 \cap W_2 \cap \cdots \cap W_N \cap U$ contains x and intersects only finitely many elements of $\mathcal{C}$. That is, $\mathcal{C}$ is locally finite. Therefore, $\mathcal{C}$ is a locally finite covering of X (though the elements of $\mathcal{C}$ may not be open or closed) and (2) follows.

Theorem 41.3 (continued 3)

Proof (continued). (2)⇒(3). Let $\mathcal{A}$ be an open covering of $\mathcal{A}$. Let $\mathcal{B}$ be the collection of all open sets U or X such that $\overline{U}$ is contained in an element of $\mathcal{A}$. So $\mathcal{B}$ is a refinement of $\mathcal{A}$. Since X is regular by hypothesis then, by lemma 31.1(a), $\mathcal{B}$ is an open cover of X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres). There is a refinement $\mathcal{C}$ of $\mathcal{B}$ that covers X and is locally finite by hypothesis (2). Let $\mathcal{D} = \{\overline{C} \mid C \in \mathcal{C}\}$. That $\mathcal{D}$ also covers X and of course the elements of $\mathcal{D}$ are closed. By Lemma 39.1(b), $\mathcal{D}$ is locally finite. Since $\mathcal{B}$ refines $\mathcal{A}$, $\mathcal{C}$ refines $\mathcal{B}$, and any $U \in \mathcal{B}$ satisfies $\overline{U} \in A$ for some $A \in \mathcal{A}$, then $\mathcal{D}$ refines $\mathcal{A}$. So $\mathcal{D}$ is a closed covering of X which is locally finite and refines $\mathcal{A}$. That is, (3) holds.

Theorem 41.3 (continued 4)

Proof (continued). (3)⇒(4). Let $\mathcal{A}$ be an open covering of X. There is a refinement $\mathcal{B}$ of $\mathcal{A}$ that covers X and is locally finite by hypothesis (3). Covering $\mathcal{B}$ is closed by (3), but we do not need this property. We now slightly "expand" each element of $\mathcal{B}$ to produce an open set in such a way that $\mathcal{B}$ is still locally finite.

For any $x \in X$, there is a neighborhood of x that intersects only finitely many elements of $\mathcal{B}$ since $\mathcal{B}$ is locally finite. So the collection of all open sets that intersect only finitely many elements of $\mathcal{B}$ is thus an open covering of X. By hypothesis (3), there is a closed refinement $\mathcal{C}$ of this new open covering that covers and is locally finite. By construction, each element of $\mathcal{C}$ intersects only finitely many elements of $\mathcal{B}$.

For each $B \in \mathcal{B}$ let $\mathcal{B}(B) = \{C \mid C \in \mathcal{C} \text{ and } C \subset X \setminus B\}$ and define $E(B) = X \setminus \cup_{C \in \mathcal{C}(B)} C$. Because $\mathcal{C}$ is locally finite collection of closed sets, the union of the elements of any subcollection of $\mathcal{C}$ is closed by Lemma 39.1 parts (a) (for the subcollection claim) and (c) (for this closed claim).

Theorem 41.3 (continued 5)

Proof (continued). So $\cup_{C \in \mathcal{C}(B)} C$ is closed and $E(B)$ is open. By definition, $B \subset E(B)$ (since $C \cap B = \varnothing$ for each $C \in \mathcal{C}(B)$).

For each $B \in \mathcal{B}$, there is $F(B) \in \mathcal{A}$ containing B since $\mathcal{B}$ is a refinement of $\mathcal{A}$. Define

$$\mathcal{D} = \{E(B) \cap F(B) \mid B \in \mathcal{B}\} = \{(X \setminus \cup_{C \in \mathcal{C}(B)} C) \cap F(B) \mid B \in \mathcal{B}\}$$

where $\mathcal{C}(B) = \{C \mid C \in \mathcal{C} \text{ and } C \subset X \setminus B$. Then $\mathcal{D}$ is a refinement of $\mathcal{A}$ since each element of $\mathcal{D}$ satisfies $E(B) \cap F(B) \subset F(B) \in \mathcal{A}$. Because $B \subset E(B) \cap F(B)$ and $\mathcal{B}$ covers X, the collection $\mathcal{D}$ covers X. Since $E(B)$ and $F(B)$ are open then $\mathcal{D}$ is an open cover of X.

Now we show that $\mathcal{D}$ is locally finite. Let $x \in X$ be given. Since $\mathcal{C}$ is locally finite, there is a neighborhood W of x that intersects only finitely many elements of $\mathcal{C}$, say $C_1, C_2, \ldots, C_k$. Because $\mathcal{C}$ covers X, open set W is covered by $C_1, C_2, \ldots, C_k$. Now if $C \in \mathcal{C}$ intersects $E(B) \cap F(B)$, then it intersects $E(B)$.

Theorem 41.3 (continued 6)

Proof (continued). Now $E(B)$ is by definition the complement of the union of all elements of $\mathcal{C}$ which do *not* intersect B. So if C intersects $E(B)$ then it must also intersect B (i.e., C cannot not intersect B!). Since C intersects only finitely many $B \in \mathcal{B}$ then C can intersect finitely many (corresponding) $E(B)$ and hence C intersects at most the same number of elements $E(B) \cap F(B)$ of $\mathcal{D}$. So neighborhood W or x intersects $C_1, C_2, \ldots, C_k$ and each of these C_i intersect finitely many elements of $\mathcal{D}$. So $\mathcal{D}$ is locally finite. Therefore, $\mathcal{D}$ is an open covering of X that is locally finite and a refinement of $\mathcal{A}$. Hence (4) follows. □

Theorem 41.4)

Theorem 41.4. Every metrizable space is paracompact.

Proof. Let X be a metrizable space. By Theorem 39.2, every open covering $\mathcal{A}$ of X has an open refinement that covers X and is countablyu locally finite (an example of an open covering is $\mathcal{A} = \{X\}$). By Lemma 41.3 (the (1)⇒(4) part) there is a refinement of $\mathcal{A}$ that covers X and is locally finite. So, by definition, X is paracompact. □

Theorem 41.5

Theorem 41.5. Every regular Lindelöf space is paracompact.

Proof. Let X be regular and Lindelöf. Since X is Lindelöf, by definition, every open covering $\mathcal{A}$ of X has a countable open subcovering of X. Trivially, this subcovering is countably locally finite (write the countable covering as a countable union of the sets consisting of single elements of the subcovering). By Lemma 41.3 (the (1)⇒(4) part), $\mathcal{A}$ has an open refinement that covers X and is locally finite. So, by definition, X is paracompact. □

Lemma 41.6

Lemma 41.6. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha\in J}$ be an indexed family of open sets covering X. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha\in J}$ of open sets covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$.

Proof. Let $\mathcal{A}$ be the collection of all open sets A such that $\overline{A}$ is contained in some element of the open covering $\{U + \alpha\}_{\alpha\in J}$. By Theorem 4.1., X is normal and so also regular (every normal space is regular) and so by Lemma 31.1(a), $\mathcal{A}$ overs X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres). Since X is paracompact then (by definition) we can find a locally finite collection $\mathcal{B}$ of open sets covering X that refines $\mathcal{A}$. Let K be an indexing set for $\mathcal{B}$, so that $\mathcal{B} = \{B_\beta\}_{\beta\in K}$ is a locally finite indexed family.

Theorem 41.6 (continued 1)

Proof (continued). Since $\mathcal{A}$ refines $\{U_\alpha\}_{\alpha\in J}$ where each $A \in \mathcal{A}$ satisfies $\overline{A} \subset U_\alpha$ for some $\alpha \in J$, and $\mathcal{B}$ refines $\mathcal{A}$, then for each $B_\beta \in \mathcal{B}$ we have that $\overline{B}_\beta \subset U_\gamma$ for some $U_\gamma \in \{U_\alpha\}_{\alpha\in J}$ and some $\gamma \in J$. Define $f : K \to J$ as $f(\beta) = \gamma$ (notice that there may be multiple choices for $f(\beta)$ here so we seems to be using the Axiom of Choice here!). For each $\alpha \in J$, define V_α to be the union of the elements in the collection $\mathcal{B}_\alpha = \{B_\beta \mid f(\beta) = \alpha\}$. So each V_α is open. For each $B_\beta \in \mathcal{B}_\alpha$ we have $B_\beta \subset U_\alpha$ (by the definition of f). Since $\mathcal{B}_\alpha \subset \mathcal{B}$ then $\mathcal{B}_\alpha$ is locally finite, and so $\overline{V}_\alpha$ equals the union of the closures of the elements of $\mathcal{B}_\alpha$ by Lemma 39.1(c). Therefore, $\overline{V}_\alpha \subset U_\alpha$.

Lemma 41.6 (continued 2)

Lemma 41.6. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha\in J}$ be an indexed family of open sets covering X. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha\in J}$ of open sets covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$.

Proof (continued). Given $x \in X$, choose a neighborhood W of x such that W intersects B_β for only finitely many values of β, say $\beta_1, \beta_2, \ldots, \beta_k$ (which is the case since $\mathcal{B}_\alpha$ is locally finite). Then W can intersect V_α only if α is one of the indices $f(\beta_1), f(\beta_2), \ldots, f(\beta_k)$ since V_α is the union of all B_β such that $f(\beta) = \alpha$. Therefore $\{V_\alpha\}_{\alpha\in J}$ is a locally finite family of open sets covering X (since $\mathcal{B} = \{B_\beta\}_{\beta\in K}$ is a covering of X and $\cup_{\beta\in K} B_\beta = \cup_{\alpha\in J} V_\alpha$) with $\overline{V}_\alpha \subset U_\alpha$, as desired. □

Theorem 41.7

Theorem 41.7. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha\in J}$ be an indexed open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}_{\alpha\in J}$.

Proof. By Lemma 41.6, since X is paracompact and Hausdorff, there is a locally finite indexed family of open sets $\{V_\alpha\}_{\alpha\in J}$ covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$. Similarly, by Lemma 41.6 as applied to open covering $\{V_\alpha\}_{\alpha\in J}$ of X, there is a locally finite indexed family of open sets $\{W_\alpha\}_{\alpha\in J}$ covering X such that $\overline{W}_\alpha \subset V_\alpha$ for all $\alpha \in J$. Next, by Theorem 41.1, X is normal. Since for each $\alpha \in J$, $\overline{W}_\alpha$ and $X \setminus V_\alpha$ are disjoint closed sets, then by Urysohn's Lemma (Theorem 33.1), there is a continuous function $\psi_\alpha : X \to [0,1]$ such that $\psi_\alpha(\overline{W}_\alpha) = \{1\}$ and $\psi_\alpha(X \setminus V_\alpha) = \{0\}$. Since ψ_α is nonzero only at points of V_α, we have $\text{Support}(\psi_\alpha) \subset \overline{V}_\alpha \subset U_\alpha$. Furthermore, the indexed family $\{\overline{V}_\alpha\}_{\alpha\in J}$ is locally finite because an open set (and so a neighborhood of some point) intersects $\overline{V}_\alpha$ only if it intersects V_α (since $\overline{V}_\alpha$ consists of the points in V_α and the limit points of V_α by Theorem 17.6).

Lemma 41.7 (continued)

Proof (continued). Hence the indexed family $\{\text{Support}(\psi_\alpha)\}_{\alpha\in J}$ is also locally finite. Note that because $\{W_\alpha\}_{\alpha\in J}$ covers X, then for any given $x \in X$ we have $x \in W_\alpha$ for some $\alpha \in J$ and so $\psi_\alpha(x) = 1$.

So for any $x \in X$ there is a neighborhood W_x of x that intersects $\text{Support}(\psi_\alpha)$ for only finitely many $\alpha \in J$ (since $\{\text{Support}(\psi_\alpha)\}_{\alpha\in J}$ is locally finite), so we interpret $\sum_{\alpha\in J}\psi_\alpha(x)$ as the sum over these finite number of $\alpha \in J$. As such, define $\Psi(x) = \sum_{\alpha\in J}\psi_\alpha(x)$. It follows that the restriction of Ψ to W_x if a finite sum of continuous (real valued) functions and so is continuous. So by Theorem 18.2(f), Ψ is continuous on X. Also, Ψ is positive (in fact, it is natural number valued), so define $\varphi_\alpha(x) = \psi_\alpha(x)/\Psi(x)$. Then

(1) $\text{Support}(\varphi_\alpha) = \text{Support}(\psi_\alpha) \subset U_\alpha$ for all $\alpha \in J$,

(2) $\text{Support}(\varphi_\alpha) = \text{Support}(\psi_\alpha)$ is locally finite, and

(3) $\sum_{\alpha\in J}\varphi_\alpha(x) = \sum_{\alpha\in J}\psi_\alpha(x)/\Psi_\alpha(x) = 1$.

That is, $\{\psi_\alpha\}_{\alpha\in J}$ is (by definition) a partition of unity dominated by $\{U_\alpha\}_{\alpha\in J}$. □

Theorem 41.8

Theorem 41.8. Let X be a paracompact Hausdorff space. Let $\mathcal{C}$ be a collection of subsets of X and for each $X \in \mathcal{C}$ let $\varepsilon_C > 0$. If $\mathcal{C}$ is locally finite, then there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) > 0$ for all x, and $f(x) \le \varepsilon_C$ for $x \in C$.

Proof. Since $\mathcal{C}$ is locally finite then (by definition) for each $x \in X$ there is a neighborhood of x which intersects only finitely many elements of $\mathcal{C}$, so create an open covering of X with such neighborhoods and denote it $\{U_\alpha\}_{\alpha\in J}$. By Theorem 41.7, there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in J}$ on X dominated by $\{U_\alpha\}_{\alpha\in J}$.

For a given $\alpha \in J$, let δ_α be the minimum of the $\varepsilon_C > 0$ as C ranges over the elements of $\mathcal{C}$ which intersect the support of φ_α (by definition of "partition of unity," $\text{Support}(\varphi_\alpha) \subset U_\alpha$ and by construction U_α intersects only finitely many $C \in \mathcal{C}$, so there are finitely many such C). If there are so such $C \in \mathcal{C}$, then set $\delta_\alpha = 1$.

Theorem 41.8 (continued)

Proof (continued). Define $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ (for given $x \in X$, this is nonzero for only finitely many $\alpha \in J$). Since $\varphi_\alpha(x) : X \to [0,1]$, $\varphi_\alpha(x) > 0$ for some $\alpha \in J$ and for such α we have $\delta_\alpha > 0$, then f is positive valued for all $x \in X$, as claimed. If $x \notin \text{Support}(\varphi_\alpha)$ then $\varphi_\alpha(x) = 0$; if $x \in \text{Support}(\varphi_\alpha)$ and $x \in C$ then $\delta_\alpha \le \varepsilon_C$. So for any $x \in C$ we have $\delta_\alpha \varphi_\alpha(x) \le \varepsilon_C \varphi_\alpha(x) \le \varepsilon_C$ for arbitrary $\alpha \in J$ and since, $\sum_{\alpha \in J} \varphi_\alpha(x) = 1$,

$$f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha \le \sum_{\alpha \in J} \varepsilon_C \varphi_\alpha(x) = \varepsilon_C \sum_{\alpha \in J} \varphi_\alpha = \varepsilon_C,$$

as claimed.

Finally, $\{\text{Support}(\varphi_\alpha)\}_{\alpha \in J}$ is locally finite (by the definition of "partition of unity") so for any $x \in X$, there is a neighborhood W of x such that W intersects only finitely many Support(φ_α)'s. So on W, $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ is the sum of finitely many continuous (real valued) functions and so f is continuous on W; that is, f restricted to each such W is continuous. By by Theorem 18.2(f), f is continuous on X, as claimed. □