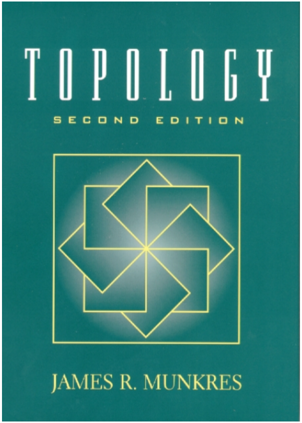
Introduction to Topology

Chapter 6. Metrization Theorems and Paracompactness

Section 41. Paracompactness—Proofs of Theorems



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Proof (continued). Let $V = \cup_{D \in \mathcal{D}} D$; then V is open in X and $B \subset V$ (since $\mathcal{D}$ covers B). Since $\mathcal{C}$ is a locally finite covering of X then $\mathcal{D}$ is a locally finite covering of B. Therefore, by Lemma 39.1(c), $\overline{V} = \cup_{D \in \mathcal{D}} \overline{D}$. Since $a \notin \overline{U}_b$ and $\overline{D} \subset \overline{U}_b$, then $a \notin \overline{V}$. Since $a \notin \overline{V}$, then a is neither in V nor is a a limit point of V (see Theorem 17.6), so there is some open set U containing a such that $U \cap V = \varnothing$. So U and V are open, $U \cap V = \varnothing$, closed set $B \subset V$ and $a \in U$. That is, X is a regular topological space.

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Now for normality, let A and B be closed sets in X. Since X is regular by the previous argument, for each $b \in B$ there are open sets U_b and $U_{b,A}$ with $b \in U_b$, $A \subset U_{b,A}$, and $U_b \cap U_{b,A} = \varnothing$. Then $A \cap \overline{U}_b = \varnothing$ ($U_{b,A}$ consists of the points in $U_{b,A}$ and the limit points of $U_{b,A}$ by Theorem 17.6;

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Proof (continued). Form the subcollection $\mathcal{D}$ of $\mathcal{C}$ consisting of every element of $\mathcal{C}$ that intersects B. Then $\mathcal{D}$ covers B (since $\mathcal{C}$ covers X). Furthermore, if $D \in \mathcal{D}$ then $\overline{D} \cap A = \varnothing$ since D intersects B and so it lies in some U_b (since $\mathcal{D} \subset \mathcal{C}$ and $\mathcal{C}$ is a refinement of $\mathcal{A}$) where $A \cap \overline{U}_b = \varnothing$ and $\overline{D} \subset \overline{U}_b$. Let $V = \cup_{D \in \mathcal{D}} D$; then V is open in X and $B \subset V$ (since $\mathcal{D}$ covers B). Since $\mathcal{C}$ is a locally finite covering of X then $\mathcal{D}$ is a locally finite covering of B.

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Proof. Let Y be a closed subspace of the paracompact space X. Let $\mathcal{A}$ be a covering of Y by sets open in Y.

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Lemma 41.3

Lemma 41.3. Let X be a regular topological space. The following conditions on X are equivalent. Every open covering of X has a refinement that is:

(1) an open covering of X and countably locally finite,

(2) a covering of X and locally finite,

(3) a closed covering of X and locally finite, and

(4) an open covering of X and locally finite (that is, X is paracompact).

Proof. (4)$\Rightarrow$(1). Since an open covering of X is countably locally finite (by definition) if it can be written as a countable union of collections of sets each of which is locally finite, then (4)$\Rightarrow$(1).

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Proof (continued). (1)⇒(2). Let $\mathcal{A}$ be an open covering of X and let $\mathcal{B}$ be an open refinement of $\mathcal{A}$ that covers X and is countably locally finite (which exists by (1)). Let $\mathcal{B} = \cup_{n=1}^{\infty} \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite (but notice that the $\mathcal{B}_n$'s may not cover X). For $i \in \mathbb{N}$, let $V_i = \cup_{U \in \mathcal{B}_i} U$. For each $n \in \mathbb{N}$ and each $U \in \mathcal{B}_n$, define $S_n(U) = U \setminus \cup_{i<n} V_i$. Let $\mathcal{C}_n = \{S_n(U) \mid U \in \mathcal{B}_n\}$.

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Let $x \in X$. Let N be the smallest index such that $x \in \mathcal{B}_N$ (since $\mathcal{B}$ is a covering of X, such N exists). Let $U_x \in \mathcal{B}_N$ contain x.

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Let $x \in X$. Let N be the smallest index such that $x \in \mathcal{B}_N$ (since $\mathcal{B}$ is a covering of X, such N exists). Let $U_x \in \mathcal{B}_N$ contain x. Since $x \notin \mathcal{B}_i$ for $i < N$, then $x \in S_N(U_x) \in \mathcal{C}_N \subset \mathcal{C}$. So $\mathcal{C}$ is a covering of X.

Theorem 41.3 (continued 1)

Proof (continued). (1)⇒(2). Let $\mathcal{A}$ be an open covering of X and let $\mathcal{B}$ be an open refinement of $\mathcal{A}$ that covers X and is countably locally finite (which exists by (1)). Let $\mathcal{B} = \cup_{n=1}^{\infty}\mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite (but notice that the $\mathcal{B}_n$'s may not cover X). For $i \in \mathbb{N}$, let $V_i = \cup_{U \in \mathcal{B}_i} U$. For each $n \in \mathbb{N}$ and each $U \in \mathcal{B}_n$, define $S_n(U) = U \setminus \cup_{i<n} V_i$. Let $\mathcal{C}_n = \{S_n(U) \mid U \in \mathcal{B}_n\}$. Then $\mathcal{C}_n$ is a refinement of $\mathcal{B}_n$ since $S_n(U) \subset U$ for each $U \in \mathcal{B}_n$ (but $S_n(U)$ may not be open [nor closed]). Let $\mathcal{C} = \sup_{n=1}^{\infty} \mathcal{C}_n$. We claim that $\mathcal{C}$ is the required locally finite refinement of $\mathcal{A}$ covering X. Since each $\mathcal{C}_n$ is a refinement of each $\mathcal{B}_n$, then $\mathcal{C}$ is a refinement of $\mathcal{B}$ and hence of $\mathcal{A}$.

Let $x \in X$. Let N be the smallest index such that $x \in \mathcal{B}_N$ (since $\mathcal{B}$ is a covering of X, such N exists). Let $U_x \in \mathcal{B}_N$ contain x. Since $x \notin \mathcal{B}_i$ for $i < N$, then $x \in S_N(U_x) \in \mathcal{C}_N \subset \mathcal{C}$. So $\mathcal{C}$ is a covering of X.

Theorem 41.3 (continued 2)

Proof (continued). Next (to show that $\mathcal{C}$ is locally finite) since each collection $\mathcal{B}_n$ is locally finite, then for each index volume $n = 1, 2, \ldots, N$ there is a neighborhood W_n of x that intersects only finitely many elements of $\mathcal{B}_n$. Now for a given $V \in \mathcal{B}_n$, if W_n intersects $S_n(V) \in \mathcal{C}_n$ then W_n must intersect $V \in \mathcal{B}_n$ since $S_n(V) \subset V$, or by the contrapositive, if W_n does not intersect $V \in \mathcal{B}_n$ then W_n does not intersect $S_n(V) \in \mathcal{C}_n$. Since W_n intersects only finitely many elements of $\mathcal{B}_n$ then W_n intersects only finitely many elements of $\mathcal{C}_n$. Since $U_x \in \mathcal{B}_M$ (the U_x containing x introduced in the previous paragraph), then U_x intersects no element of $\mathcal{C}_n$ for $n > N$ (since $\mathcal{C}_n = \{S_n(U) \mid U \in \mathcal{B}_n\}$ and $S_n(U) = U \setminus \cup_{i<n} V_i$). So the open set $W_1 \cap W_2 \cap \cdots \cap W_N \cap U$ contains x and intersects only finitely many elements of $\mathcal{C}$.

Theorem 41.3 (continued 2)

Proof (continued). Next (to show that $\mathcal{C}$ is locally finite) since each collection $\mathcal{B}_n$ is locally finite, then for each index volume $n = 1, 2, \ldots, N$ there is a neighborhood W_n of x that intersects only finitely many elements of $\mathcal{B}_n$. Now for a given $V \in \mathcal{B}_n$, if W_n intersects $S_n(V) \in \mathcal{C}_n$ then W_n must intersect $V \in \mathcal{B}_n$ since $S_n(V) \subset V$, or by the contrapositive, if W_n does not intersect $V \in \mathcal{B}_n$ then W_n does not intersect $S_n(V) \in \mathcal{C}_n$. Since W_n intersects only finitely many elements of $\mathcal{B}_n$ then W_n intersects only finitely many elements of $\mathcal{C}_n$. Since $U_x \in \mathcal{B}_M$ (the U_x containing x introduced in the previous paragraph), then U_x intersects no element of $\mathcal{C}_n$ for $n > N$ (since $\mathcal{C}_n = \{S_n(U) \mid U \in \mathcal{B}_n\}$ and $S_n(U) = U \setminus \cup_{i<n} V_i$). So the open set $W_1 \cap W_2 \cap \cdots \cap W_N \cap U$ contains x and intersects only finitely many elements of $\mathcal{C}$. That is, $\mathcal{C}$ is locally finite. Therefore, $\mathcal{C}$ is a locally finite covering of X (though the elements of $\mathcal{C}$ may not be open or closed) and (2) follows.

Theorem 41.3 (continued 2)

Proof (continued). Next (to show that $\mathcal{C}$ is locally finite) since each collection $\mathcal{B}_n$ is locally finite, then for each index volume $n = 1, 2, \ldots, N$ there is a neighborhood W_n of x that intersects only finitely many elements of $\mathcal{B}_n$. Now for a given $V \in \mathcal{B}_n$, if W_n intersects $S_n(V) \in \mathcal{C}_n$ then W_n must intersect $V \in \mathcal{B}_n$ since $S_n(V) \subset V$, or by the contrapositive, if W_n does not intersect $V \in \mathcal{B}_n$ then W_n does not intersect $S_n(V) \in \mathcal{C}_n$. Since W_n intersects only finitely many elements of $\mathcal{B}_n$ then W_n intersects only finitely many elements of $\mathcal{C}_n$. Since $U_x \in \mathcal{B}_M$ (the U_x containing x introduced in the previous paragraph), then U_x intersects no element of $\mathcal{C}_n$ for $n > N$ (since $\mathcal{C}_n = \{S_n(U) \mid U \in \mathcal{B}_n\}$ and $S_n(U) = U \setminus \cup_{i<n} V_i$). So the open set $W_1 \cap W_2 \cap \cdots \cap W_N \cap U$ contains x and intersects only finitely many elements of $\mathcal{C}$. That is, $\mathcal{C}$ is locally finite. Therefore, $\mathcal{C}$ is a locally finite covering of X (though the elements of $\mathcal{C}$ may not be open or closed) and (2) follows.

Theorem 41.3 (continued 3)

Proof (continued). (2)$\Rightarrow$(3). Let $\mathcal{A}$ be an open covering of $\mathcal{A}$. Let $\mathcal{B}$ be the collection of all open sets U or X such that $\overline{U}$ is contained in an element of $\mathcal{A}$. So $\mathcal{B}$ is a refinement of $\mathcal{A}$. Since X is regular by hypothesis then, by lemma 31.1(a), $\mathcal{B}$ is an open cover of X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres).

Theorem 41.3 (continued 3)

Proof (continued). (2)⇒(3). Let $\mathcal{A}$ be an open covering of $\mathcal{A}$. Let $\mathcal{B}$ be the collection of all open sets U or X such that $\overline{U}$ is contained in an element of $\mathcal{A}$. So $\mathcal{B}$ is a refinement of $\mathcal{A}$. Since X is regular by hypothesis then, by lemma 31.1(a), $\mathcal{B}$ is an open cover of X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres). There is a refinement $\mathcal{C}$ of $\mathcal{B}$ that covers X and is locally finite by hypothesis (2). Let $\mathcal{D} = \{\overline{C} \mid C \in \mathcal{C}\}$. That $\mathcal{D}$ also covers X and of course the elements of $\mathcal{D}$ are closed. By Lemma 39.1(b), $\mathcal{D}$ is locally finite.

Theorem 41.3 (continued 3)

Proof (continued). (2)⇒(3). Let $\mathcal{A}$ be an open covering of $\mathcal{A}$. Let $\mathcal{B}$ be the collection of all open sets U or X such that $\overline{U}$ is contained in an element of $\mathcal{A}$. So $\mathcal{B}$ is a refinement of $\mathcal{A}$. Since X is regular by hypothesis then, by lemma 31.1(a), $\mathcal{B}$ is an open cover of X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres). There is a refinement $\mathcal{C}$ of $\mathcal{B}$ that covers X and is locally finite by hypothesis (2). Let $\mathcal{D} = \{\overline{C} \mid C \in \mathcal{C}\}$. That $\mathcal{D}$ also covers X and of course the elements of $\mathcal{D}$ are closed. By Lemma 39.1(b), $\mathcal{D}$ is locally finite. Since $\mathcal{B}$ refines $\mathcal{A}$, $\mathcal{C}$ refines $\mathcal{B}$, and any $U \in \mathcal{B}$ satisfies $\overline{U} \in A$ for some $A \in \mathcal{A}$, then $\mathcal{D}$ refines $\mathcal{A}$. So $\mathcal{D}$ is a closed covering of X which is locally finite and refines $\mathcal{A}$. That is, (3) holds.

Theorem 41.3 (continued 3)

Proof (continued). (2)⇒(3). Let $\mathcal{A}$ be an open covering of $\mathcal{A}$. Let $\mathcal{B}$ be the collection of all open sets U or X such that $\overline{U}$ is contained in an element of $\mathcal{A}$. So $\mathcal{B}$ is a refinement of $\mathcal{A}$. Since X is regular by hypothesis then, by lemma 31.1(a), $\mathcal{B}$ is an open cover of X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres). There is a refinement $\mathcal{C}$ of $\mathcal{B}$ that covers X and is locally finite by hypothesis (2). Let $\mathcal{D} = \{\overline{C} \mid C \in \mathcal{C}\}$. That $\mathcal{D}$ also covers X and of course the elements of $\mathcal{D}$ are closed. By Lemma 39.1(b), $\mathcal{D}$ is locally finite. Since $\mathcal{B}$ refines $\mathcal{A}$, $\mathcal{C}$ refines $\mathcal{B}$, and any $U \in \mathcal{B}$ satisfies $\overline{U} \in A$ for some $A \in \mathcal{A}$, then $\mathcal{D}$ refines $\mathcal{A}$. So $\mathcal{D}$ is a closed covering of X which is locally finite and refines $\mathcal{A}$. That is, (3) holds.

Theorem 41.3 (continued 4)

Proof (continued). (3)$\Rightarrow$(4). Let $\mathcal{A}$ be an open covering of X. There is a refinement $\mathcal{B}$ of $\mathcal{A}$ that covers X and is locally finite by hypothesis (3). Covering $\mathcal{B}$ is closed by (3), but we do not need this property. We now slightly "expand" each element of $\mathcal{B}$ to produce an open set in such a way that $\mathcal{B}$ is still locally finite.

For any $x \in X$, there is a neighborhood of x that intersects only finitely many elements of $\mathcal{B}$ since $\mathcal{B}$ is locally finite. So the collection of all open sets that intersect only finitely many elements of $\mathcal{B}$ is thus an open covering of X. By hypothesis (3), there is a closed refinement $\mathcal{C}$ of this new open covering that covers and is locally finite. By construction, each element of $\mathcal{C}$ intersects only finitely many elements of $\mathcal{B}$.

Theorem 41.3 (continued 4)

Proof (continued). (3)⇒(4). Let $\mathcal{A}$ be an open covering of X. There is a refinement $\mathcal{B}$ of $\mathcal{A}$ that covers X and is locally finite by hypothesis (3). Covering $\mathcal{B}$ is closed by (3), but we do not need this property. We now slightly "expand" each element of $\mathcal{B}$ to produce an open set in such a way that $\mathcal{B}$ is still locally finite.

For any $x \in X$, there is a neighborhood of x that intersects only finitely many elements of $\mathcal{B}$ since $\mathcal{B}$ is locally finite. So the collection of all open sets that intersect only finitely many elements of $\mathcal{B}$ is thus an open covering of X. By hypothesis (3), there is a closed refinement $\mathcal{C}$ of this new open covering that covers and is locally finite. By construction, each element of $\mathcal{C}$ intersects only finitely many elements of $\mathcal{B}$.

For each $B \in \mathcal{B}$ let $\mathcal{B}(B) = \{C \mid C \in \mathcal{C} \text{ and } C \subset X \setminus B\}$ and define $E(B) = X \setminus \cup_{C \in \mathcal{C}(B)} C$. Because $\mathcal{C}$ is locally finite collection of closed sets, the union of the elements of any subcollection of $\mathcal{C}$ is closed by Lemma 39.1 parts (a) (for the subcollection claim) and (c) (for this closed claim).

Theorem 41.3 (continued 4)

Proof (continued). (3)⇒(4). Let $\mathcal{A}$ be an open covering of X. There is a refinement $\mathcal{B}$ of $\mathcal{A}$ that covers X and is locally finite by hypothesis (3). Covering $\mathcal{B}$ is closed by (3), but we do not need this property. We now slightly "expand" each element of $\mathcal{B}$ to produce an open set in such a way that $\mathcal{B}$ is still locally finite.

For any $x \in X$, there is a neighborhood of x that intersects only finitely many elements of $\mathcal{B}$ since $\mathcal{B}$ is locally finite. So the collection of all open sets that intersect only finitely many elements of $\mathcal{B}$ is thus an open covering of X. By hypothesis (3), there is a closed refinement $\mathcal{C}$ of this new open covering that covers and is locally finite. By construction, each element of $\mathcal{C}$ intersects only finitely many elements of $\mathcal{B}$.

For each $B \in \mathcal{B}$ let $\mathcal{B}(B) = \{C \mid C \in \mathcal{C} \text{ and } C \subset X \setminus B\}$ and define $E(B) = X \setminus \cup_{C \in \mathcal{C}(B)} C$. Because $\mathcal{C}$ is locally finite collection of closed sets, the union of the elements of any subcollection of $\mathcal{C}$ is closed by Lemma 39.1 parts (a) (for the subcollection claim) and (c) (for this closed claim).

Theorem 41.3 (continued 5)

Proof (continued). So $\cup_{C \in \mathcal{C}(B)} C$ is closed and $E(B)$ is open. By definition, $B \subset E(B)$ (since $C \cap B = \varnothing$ for each $C \in \mathcal{C}(B)$).

For each $B \in \mathcal{B}$, there is $F(B) \in \mathcal{A}$ containing B since $\mathcal{B}$ is a refinement of $\mathcal{A}$. Define

$$\mathcal{D} = \{E(B) \cap F(B) \mid B \in \mathcal{B}\} = \{(X \setminus \cup_{C \in \mathcal{C}(B)} C) \cap F(B) \mid B \in \mathcal{B}\}$$

where $\mathcal{C}(B) = \{C \mid C \in \mathcal{C} \text{ and } C \subset X \setminus B$.

Theorem 41.3 (continued 5)

Proof (continued). So $\cup_{C \in \mathcal{C}(B)} C$ is closed and $E(B)$ is open. By definition, $B \subset E(B)$ (since $C \cap B = \varnothing$ for each $C \in \mathcal{C}(B)$).

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where $\mathcal{C}(B) = \{C \mid C \in \mathcal{C}$ and $C \subset X \setminus B$. Then $\mathcal{D}$ is a refinement of $\mathcal{A}$ since each element of $\mathcal{D}$ satisfies $E(B) \cap F(B) \subset F(B) \in \mathcal{A}$. Because $B \subset E(B) \cap F(B)$ and $\mathcal{B}$ covers X, the collection $\mathcal{D}$ covers X. Since $E(B)$ and $F(B)$ are open then $\mathcal{D}$ is an open cover of X.

Theorem 41.3 (continued 5)

Proof (continued). So $\cup_{C \in \mathcal{C}(B)} C$ is closed and $E(B)$ is open. By definition, $B \subset E(B)$ (since $C \cap B = \varnothing$ for each $C \in \mathcal{C}(B)$).

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where $\mathcal{C}(B) = \{C \mid C \in \mathcal{C} \text{ and } C \subset X \setminus B$. Then $\mathcal{D}$ is a refinement of $\mathcal{A}$ since each element of $\mathcal{D}$ satisfies $E(B) \cap F(B) \subset F(B) \in \mathcal{A}$. Because $B \subset E(B) \cap F(B)$ and $\mathcal{B}$ covers X, the collection $\mathcal{D}$ covers X. Since $E(B)$ and $F(B)$ are open then $\mathcal{D}$ is an open cover of X.

Now we show that $\mathcal{D}$ is locally finite. Let $x \in X$ be given. Since $\mathcal{C}$ is locally finite, there is a neighborhood W of x that intersects only finitely many elements of $\mathcal{C}$, say $C_1, C_2, \ldots, C_k$.

Theorem 41.3 (continued 5)

Proof (continued). So $\cup_{C \in \mathcal{C}(B)} C$ is closed and $E(B)$ is open. By definition, $B \subset E(B)$ (since $C \cap B = \varnothing$ for each $C \in \mathcal{C}(B)$).

For each $B \in \mathcal{B}$, there is $F(B) \in \mathcal{A}$ containing B since $\mathcal{B}$ is a refinement of $\mathcal{A}$. Define

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where $\mathcal{C}(B) = \{C \mid C \in \mathcal{C}$ and $C \subset X \setminus B$. Then $\mathcal{D}$ is a refinement of $\mathcal{A}$ since each element of $\mathcal{D}$ satisfies $E(B) \cap F(B) \subset F(B) \in \mathcal{A}$. Because $B \subset E(B) \cap F(B)$ and $\mathcal{B}$ covers X, the collection $\mathcal{D}$ covers X. Since $E(B)$ and $F(B)$ are open then $\mathcal{D}$ is an open cover of X.

Now we show that $\mathcal{D}$ is locally finite. Let $x \in X$ be given. Since $\mathcal{C}$ is locally finite, there is a neighborhood W of x that intersects only finitely many elements of $\mathcal{C}$, say $C_1, C_2, \ldots, C_k$. Because $\mathcal{C}$ covers X, open set W is covered by $C_1, C_2, \ldots, C_k$. Now if $C \in \mathcal{C}$ intersects $E(B) \cap F(B)$, then it intersects $E(B)$.

Theorem 41.3 (continued 5)

Proof (continued). So $\cup_{C \in \mathcal{C}(B)} C$ is closed and $E(B)$ is open. By definition, $B \subset E(B)$ (since $C \cap B = \varnothing$ for each $C \in \mathcal{C}(B)$).

For each $B \in \mathcal{B}$, there is $F(B) \in \mathcal{A}$ containing B since $\mathcal{B}$ is a refinement of $\mathcal{A}$. Define

$$\mathcal{D} = \{E(B) \cap F(B) \mid B \in \mathcal{B}\} = \{(X \setminus \cup_{C \in \mathcal{C}(B)} C) \cap F(B) \mid B \in \mathcal{B}\}$$

where $\mathcal{C}(B) = \{C \mid C \in \mathcal{C}$ and $C \subset X \setminus B$. Then $\mathcal{D}$ is a refinement of $\mathcal{A}$ since each element of $\mathcal{D}$ satisfies $E(B) \cap F(B) \subset F(B) \in \mathcal{A}$. Because $B \subset E(B) \cap F(B)$ and $\mathcal{B}$ covers X, the collection $\mathcal{D}$ covers X. Since $E(B)$ and $F(B)$ are open then $\mathcal{D}$ is an open cover of X.

Now we show that $\mathcal{D}$ is locally finite. Let $x \in X$ be given. Since $\mathcal{C}$ is locally finite, there is a neighborhood W of x that intersects only finitely many elements of $\mathcal{C}$, say $C_1, C_2, \ldots, C_k$. Because $\mathcal{C}$ covers X, open set W is covered by $C_1, C_2, \ldots, C_k$. Now if $C \in \mathcal{C}$ intersects $E(B) \cap F(B)$, then it intersects $E(B)$.

Theorem 41.3 (continued 6)

Proof (continued). Now $E(B)$ is by definition the complement of the union of all elements of $\mathcal{C}$ which do *not* intersect B. So if C intersects $E(B)$ then it must also intersect B (i.e., C cannot not intersect B!). Since C intersects only finitely many $B \in \mathcal{B}$ then C can intersect finitely many (corresponding) $E(B)$ and hence C intersects at most the same number of elements $E(B) \cap F(B)$ of $\mathcal{D}$. So neighborhood W or x intersects $C_1, C_2, \ldots, C_k$ and each of these C_i intersect finitely many elements of $\mathcal{D}$. So $\mathcal{D}$ is locally finite.

Theorem 41.3 (continued 6)

Proof (continued). Now $E(B)$ is by definition the complement of the union of all elements of $\mathcal{C}$ which do *not* intersect B. So if C intersects $E(B)$ then it must also intersect B (i.e., C cannot not intersect B!). Since C intersects only finitely many $B \in \mathcal{B}$ then C can intersect finitely many (corresponding) $E(B)$ and hence C intersects at most the same number of elements $E(B) \cap F(B)$ of $\mathcal{D}$. So neighborhood W or x intersects $C_1, C_2, \ldots, C_k$ and each of these C_i intersect finitely many elements of $\mathcal{D}$. So $\mathcal{D}$ is locally finite. Therefore, $\mathcal{D}$ is an open covering of X that is locally finite and a refinement of $\mathcal{A}$. Hence (4) follows. □

Theorem 41.3 (continued 6)

Proof (continued). Now $E(B)$ is by definition the complement of the union of all elements of $\mathcal{C}$ which do *not* intersect B. So if C intersects $E(B)$ then it must also intersect B (i.e., C cannot not intersect B!). Since C intersects only finitely many $B \in \mathcal{B}$ then C can intersect finitely many (corresponding) $E(B)$ and hence C intersects at most the same number of elements $E(B) \cap F(B)$ of $\mathcal{D}$. So neighborhood W or x intersects $C_1, C_2, \ldots, C_k$ and each of these C_i intersect finitely many elements of $\mathcal{D}$. So $\mathcal{D}$ is locally finite. Therefore, $\mathcal{D}$ is an open covering of X that is locally finite and a refinement of $\mathcal{A}$. Hence (4) follows. □

Theorem 41.4)

Theorem 41.4. Every metrizable space is paracompact.

Proof. Let X be a metrizable space. By Theorem 39.2, every open covering $\mathcal{A}$ of X has an open refinement that covers X and is countablyu locally finite (an example of an open covering is $\mathcal{A} = \{X\}$).

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Theorem 41.5

Theorem 41.5. Every regular Lindelöf space is paracompact.

Proof. Let X be regular and Lindelöf. Since X is Lindelöf, by definition, every open covering $\mathcal{A}$ of X has a countable open subcovering of X. Trivially, this subcovering is countably locally finite (write the countable covering as a countable union of the sets consisting of single elements of the subcovering).

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Lemma 41.6

Lemma 41.6. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed family of open sets covering X. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha \in J}$ of open sets covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$.

Proof. Let $\mathcal{A}$ be the collection of all open sets A such that $\overline{A}$ is contained in some element of the open covering $\{U + \alpha\}_{\alpha \in J}$. By Theorem 4.1., X is normal and so also regular (every normal space is regular) and so by Lemma 31.1(a), $\mathcal{A}$ overs X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres).

Lemma 41.6

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Proof. Let $\mathcal{A}$ be the collection of all open sets A such that $\overline{A}$ is contained in some element of the open covering $\{U + \alpha\}_{\alpha \in J}$. By Theorem 4.1., X is normal and so also regular (every normal space is regular) and so by Lemma 31.1(a), $\mathcal{A}$ overs X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres). Since X is paracompact then (by definition) we can find a locally finite collection $\mathcal{B}$ of open sets covering X that refines $\mathcal{A}$. Let K be an indexing set for $\mathcal{B}$, so that $\mathcal{B} = \{B_\beta\}_{\beta \in K}$ is a locally finite indexed family.

Lemma 41.6

Lemma 41.6. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed family of open sets covering X. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha \in J}$ of open sets covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$.

Proof. Let $\mathcal{A}$ be the collection of all open sets A such that $\overline{A}$ is contained in some element of the open covering $\{U + \alpha\}_{\alpha \in J}$. By Theorem 4.1., X is normal and so also regular (every normal space is regular) and so by Lemma 31.1(a), $\mathcal{A}$ overs X (notice that in a regular space, by *definition*, one point sets are closed; see Munkres page 195 or the class notes for Section 31 where this is addressed and the "Tychonoff separation property" is mentioned though it is not in Munkres). Since X is paracompact then (by definition) we can find a locally finite collection $\mathcal{B}$ of open sets covering X that refines $\mathcal{A}$. Let K be an indexing set for $\mathcal{B}$, so that $\mathcal{B} = \{B_\beta\}_{\beta \in K}$ is a locally finite indexed family.

Theorem 41.6 (continued 1)

Proof (continued). Since $\mathcal{A}$ refines $\{U_\alpha\}_{\alpha \in J}$ where each $A \in \mathcal{A}$ satisfies $\overline{A} \subset U_\alpha$ for some $\alpha \in J$, and $\mathcal{B}$ refines $\mathcal{A}$, then for each $B_\beta \in \mathcal{B}$ we have that $\overline{B}_\beta \subset U_\gamma$ for some $U_\gamma \in \{U_\alpha\}_{\alpha \in J}$ and some $\gamma \in J$. Define $f : K \to J$ as $f(\beta) = \gamma$ (notice that there may be multiple choices for $f(\beta)$ here so we seems to be using the Axiom of Choice here!). For each $\alpha \in J$, define V_α to be the union of the elements in the collection $\mathcal{B}_\alpha = \{B_\beta \mid f(\beta) = \alpha\}$. So each V_α is open.

Theorem 41.6 (continued 1)

Proof (continued). Since $\mathcal{A}$ refines $\{U_\alpha\}_{\alpha \in J}$ where each $A \in \mathcal{A}$ satisfies $\overline{A} \subset U_\alpha$ for some $\alpha \in J$, and $\mathcal{B}$ refines $\mathcal{A}$, then for each $B_\beta \in \mathcal{B}$ we have that $\overline{B}_\beta \subset U_\gamma$ for some $U_\gamma \in \{U_\alpha\}_{\alpha \in J}$ and some $\gamma \in J$. Define $f : K \to J$ as $f(\beta) = \gamma$ (notice that there may be multiple choices for $f(\beta)$ here so we seems to be using the Axiom of Choice here!). For each $\alpha \in J$, define V_α to be the union of the elements in the collection $\mathcal{B}_\alpha = \{B_\beta \mid f(\beta) = \alpha\}$. So each V_α is open. For each $B_\beta \in \mathcal{B}_\alpha$ we have $B_\beta \subset U_\alpha$ (by the definition of f). Since $\mathcal{B}_\alpha \subset \mathcal{B}$ then $\mathcal{B}_\alpha$ is locally finite, and so $\overline{V}_\alpha$ equals the union of the closures of the elements of $\mathcal{B}_\alpha$ by Lemma 39.1(c). Therefore, $\overline{V}_\alpha \subset U_\alpha$.

Theorem 41.6 (continued 1)

Proof (continued). Since $\mathcal{A}$ refines $\{U_\alpha\}_{\alpha \in J}$ where each $A \in \mathcal{A}$ satisfies $\overline{A} \subset U_\alpha$ for some $\alpha \in J$, and $\mathcal{B}$ refines $\mathcal{A}$, then for each $B_\beta \in \mathcal{B}$ we have that $\overline{B}_\beta \subset U_\gamma$ for some $U_\gamma \in \{U_\alpha\}_{\alpha \in J}$ and some $\gamma \in J$. Define $f : K \to J$ as $f(\beta) = \gamma$ (notice that there may be multiple choices for $f(\beta)$ here so we seems to be using the Axiom of Choice here!). For each $\alpha \in J$, define V_α to be the union of the elements in the collection $\mathcal{B}_\alpha = \{B_\beta \mid f(\beta) = \alpha\}$. So each V_α is open. For each $B_\beta \in \mathcal{B}_\alpha$ we have $B_\beta \subset U_\alpha$ (by the definition of f). Since $\mathcal{B}_\alpha \subset \mathcal{B}$ then $\mathcal{B}_\alpha$ is locally finite, and so $\overline{V}_\alpha$ equals the union of the closures of the elements of $\mathcal{B}_\alpha$ by Lemma 39.1(c). Therefore, $\overline{V}_\alpha \subset U_\alpha$.

Lemma 41.6 (continued 2)

Lemma 41.6. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed family of open sets covering X. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha \in J}$ of open sets covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$.

Proof (continued). Given $x \in X$, choose a neighborhood W of x such that W intersects B_β for only finitely many values of β, say $\beta_1, \beta_2, \ldots, \beta_k$ (which is the case since $\mathcal{B}_\alpha$ is locally finite). Then W can intersect V_α only if α is one of the indices $f(\beta_1), f(\beta_2), \ldots, f(\beta_k)$ since V_α is the union of all B_β such that $f(\beta) = \alpha$. Therefore $\{V_\alpha\}_{\alpha \in J}$ is a locally finite family of open sets covering X (since $\mathcal{B} = \{B_\beta\}_{\beta \in K}$ is a covering of X and $\cup_{\beta \in K} B_\beta = \cup_{\alpha \in J} V_\alpha$) with $\overline{V}_\alpha \subset U_\alpha$, as desired. □

Lemma 41.6 (continued 2)

Lemma 41.6. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed family of open sets covering X. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha \in J}$ of open sets covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$.

Proof (continued). Given $x \in X$, choose a neighborhood W of x such that W intersects B_β for only finitely many values of β, say $\beta_1, \beta_2, \ldots, \beta_k$ (which is the case since $\mathcal{B}_\alpha$ is locally finite). Then W can intersect V_α only if α is one of the indices $f(\beta_1), f(\beta_2), \ldots, f(\beta_k)$ since V_α is the union of all B_β such that $f(\beta) = \alpha$. Therefore $\{V_\alpha\}_{\alpha \in J}$ is a locally finite family of open sets covering X (since $\mathcal{B} = \{B_\beta\}_{\beta \in K}$ is a covering of X and $\cup_{\beta \in K} B_\beta = \cup_{\alpha \in J} V_\alpha$) with $\overline{V}_\alpha \subset U_\alpha$, as desired. □

Theorem 41.7

Theorem 41.7. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}_{\alpha \in J}$.

Proof. By Lemma 41.6, since X is paracompact and Hausdorff, there is a locally finite indexed family of open sets $\{V_\alpha\}_{\alpha \in J}$ covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$. Similarly, by Lemma 41.6 as applied to open covering $\{V_\alpha\}_{\alpha \in J}$ of X, there is a locally finite indexed family of open sets $\{W_\alpha\}_{\alpha \in J}$ covering X such that $\overline{W}_\alpha \subset V_\alpha$ for all $\alpha \in J$.

Theorem 41.7

Theorem 41.7. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}_{\alpha \in J}$.

Proof. By Lemma 41.6, since X is paracompact and Hausdorff, there is a locally finite indexed family of open sets $\{V_\alpha\}_{\alpha \in J}$ covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$. Similarly, by Lemma 41.6 as applied to open covering $\{V_\alpha\}_{\alpha \in J}$ of X, there is a locally finite indexed family of open sets $\{W_\alpha\}_{\alpha \in J}$ covering X such that $\overline{W}_\alpha \subset V_\alpha$ for all $\alpha \in J$. Next, by Theorem 41.1, X is normal. Since for each $\alpha \in J$, $\overline{W}_\alpha$ and $X \setminus \overline{V}_\alpha$ are disjoint closed sets, then by Urysohn's Lemma (Theorem 33.1), there is a continuous function $\psi_\alpha : X \to [0,1]$ such that $\psi_\alpha(\overline{W}_\alpha) = \{1\}$ and $\psi_\alpha(X \setminus V_\alpha) = \{0\}$. Since ψ_α is nonzero only at points of V_α, we have Support$(\psi_\alpha) \subset \overline{V}_\alpha \subset U_\alpha$.

Theorem 41.7

Theorem 41.7. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}_{\alpha \in J}$.

Proof. By Lemma 41.6, since X is paracompact and Hausdorff, there is a locally finite indexed family of open sets $\{V_\alpha\}_{\alpha \in J}$ covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$. Similarly, by Lemma 41.6 as applied to open covering $\{V_\alpha\}_{\alpha \in J}$ of X, there is a locally finite indexed family of open sets $\{W_\alpha\}_{\alpha \in J}$ covering X such that $\overline{W}_\alpha \subset V_\alpha$ for all $\alpha \in J$. Next, by Theorem 41.1, X is normal. Since for each $\alpha \in J$, $\overline{W}_\alpha$ and $X \setminus \overline{V}_\alpha$ are disjoint closed sets, then by Urysohn's Lemma (Theorem 33.1), there is a continuous function $\psi_\alpha : X \to [0,1]$ such that $\psi_\alpha(\overline{W}_\alpha) = \{1\}$ and $\psi_\alpha(X \setminus V_\alpha) = \{0\}$. Since ψ_α is nonzero only at points of V_α, we have $\text{Support}(\psi_\alpha) \subset \overline{V}_\alpha \subset U_\alpha$. Furthermore, the indexed family $\{\overline{V}_\alpha\}_{\alpha \in J}$ is locally finite because an open set (and so a neighborhood of some point) intersects $\overline{V}_\alpha$ only if it intersects V_α (since $\overline{V}_\alpha$ consists of the points in V_α and the limit points of V_α by Theorem 17.6).

Theorem 41.7

Theorem 41.7. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}_{\alpha \in J}$.

Proof. By Lemma 41.6, since X is paracompact and Hausdorff, there is a locally finite indexed family of open sets $\{V_\alpha\}_{\alpha \in J}$ covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$. Similarly, by Lemma 41.6 as applied to open covering $\{V_\alpha\}_{\alpha \in J}$ of X, there is a locally finite indexed family of open sets $\{W_\alpha\}_{\alpha \in J}$ covering X such that $\overline{W}_\alpha \subset V_\alpha$ for all $\alpha \in J$. Next, by Theorem 41.1, X is normal. Since for each $\alpha \in J$, $\overline{W}_\alpha$ and $X \setminus \overline{V}_\alpha$ are disjoint closed sets, then by Urysohn's Lemma (Theorem 33.1), there is a continuous function $\psi_\alpha : X \to [0, 1]$ such that $\psi_\alpha(\overline{W}_\alpha) = \{1\}$ and $\psi_\alpha(X \setminus V_\alpha) = \{0\}$. Since ψ_α is nonzero only at points of V_α, we have $\text{Support}(\psi_\alpha) \subset \overline{V}_\alpha \subset U_\alpha$. Furthermore, the indexed family $\{\overline{V}_\alpha\}_{\alpha \in J}$ is locally finite because an open set (and so a neighborhood of some point) intersects $\overline{V}_\alpha$ only if it intersects V_α (since $\overline{V}_\alpha$ consists of the points in V_α and the limit points of V_α by Theorem 17.6).

Lemma 41.7 (continued)

Proof (continued). Hence the indexed family $\{\text{Support}(\psi_\alpha)\}_{\alpha \in J}$ is also locally finite. Note that because $\{W_\alpha\}_{\alpha \in J}$ covers X, then for any given $x \in X$ we have $x \in W_\alpha$ for some $\alpha \in J$ and so $\psi_\alpha(x) = 1$.

So for any $x \in X$ there is a neighborhood W_x of x that intersects $\text{Support}(\psi_\alpha)$ for only finitely many $\alpha \in J$ (since $\{\text{Support}(\psi_\alpha)\}_{\alpha \in J}$ is locally finite), so we interpret $\sum_{\alpha \in J} \psi_\alpha(x)$ as the sum over these finite number of $\alpha \in J$. As such, define $\Psi(x) = \sum_{\alpha \in J} \psi_\alpha(x)$.

Lemma 41.7 (continued)

Proof (continued). Hence the indexed family $\{\text{Support}(\psi_\alpha)\}_{\alpha \in J}$ is also locally finite. Note that because $\{W_\alpha\}_{\alpha \in J}$ covers X, then for any given $x \in X$ we have $x \in W_\alpha$ for some $\alpha \in J$ and so $\psi_\alpha(x) = 1$.

So for any $x \in X$ there is a neighborhood W_x of x that intersects $\text{Support}(\psi_\alpha)$ for only finitely many $\alpha \in J$ (since $\{\text{Support}(\psi_\alpha)\}_{\alpha \in J}$ is locally finite), so we interpret $\sum_{\alpha \in J} \psi_\alpha(x)$ as the sum over these finite number of $\alpha \in J$. As such, define $\Psi(x) = \sum_{\alpha \in J} \psi_\alpha(x)$. It follows that the restriction of Ψ to W_x if a finite sum of continuous (real valued) functions and so is continuous. So by Theorem 18.2(f), Ψ is continuous on X. Also, Ψ is positive (in fact, it is natural number valued), so define $\varphi_\alpha(x) = \psi_\alpha(x)/\Psi(x)$.

Lemma 41.7 (continued)

Proof (continued). Hence the indexed family $\{\text{Support}(\psi_\alpha)\}_{\alpha \in J}$ is also locally finite. Note that because $\{W_\alpha\}_{\alpha \in J}$ covers X, then for any given $x \in X$ we have $x \in W_\alpha$ for some $\alpha \in J$ and so $\psi_\alpha(x) = 1$.

So for any $x \in X$ there is a neighborhood W_x of x that intersects $\text{Support}(\psi_\alpha)$ for only finitely many $\alpha \in J$ (since $\{\text{Support}(\psi_\alpha)\}_{\alpha \in J}$ is locally finite), so we interpret $\sum_{\alpha \in J} \psi_\alpha(x)$ as the sum over these finite number of $\alpha \in J$. As such, define $\Psi(x) = \sum_{\alpha \in J} \psi_\alpha(x)$. It follows that the restriction of Ψ to W_x if a finite sum of continuous (real valued) functions and so is continuous. So by Theorem 18.2(f), Ψ is continuous on X. Also, Ψ is positive (in fact, it is natural number valued), so define $\varphi_\alpha(x) = \psi_\alpha(x)/\Psi(x)$. Then

(1) $\text{Support}(\varphi_\alpha) = \text{Support}(\psi_\alpha) \subset U_\alpha$ for all $\alpha \in J$,

(2) $\text{Support}(\varphi_\alpha) = \text{Support}(\psi_\alpha)$ is locally finite, and

(3) $\sum_{\alpha \in J} \varphi_\alpha(x) = \sum_{\alpha \in J} \psi_\alpha(x)/\Psi_\alpha(x) = 1$.

That is, $\{\psi_\alpha\}_{\alpha \in J}$ is (by definition) a partition of unity dominated by $\{U_\alpha\}_{\alpha \in J}$. □

Lemma 41.7 (continued)

Proof (continued). Hence the indexed family $\{\text{Support}(\psi_\alpha)\}_{\alpha \in J}$ is also locally finite. Note that because $\{W_\alpha\}_{\alpha \in J}$ covers X, then for any given $x \in X$ we have $x \in W_\alpha$ for some $\alpha \in J$ and so $\psi_\alpha(x) = 1$.

So for any $x \in X$ there is a neighborhood W_x of x that intersects $\text{Support}(\psi_\alpha)$ for only finitely many $\alpha \in J$ (since $\{\text{Support}(\psi_\alpha)\}_{\alpha \in J}$ is locally finite), so we interpret $\sum_{\alpha \in J} \psi_\alpha(x)$ as the sum over these finite number of $\alpha \in J$. As such, define $\Psi(x) = \sum_{\alpha \in J} \psi_\alpha(x)$. It follows that the restriction of Ψ to W_x if a finite sum of continuous (real valued) functions and so is continuous. So by Theorem 18.2(f), Ψ is continuous on X. Also, Ψ is positive (in fact, it is natural number valued), so define $\varphi_\alpha(x) = \psi_\alpha(x)/\Psi(x)$. Then

(1) $\text{Support}(\varphi_\alpha) = \text{Support}(\psi_\alpha) \subset U_\alpha$ for all $\alpha \in J$,
(2) $\text{Support}(\varphi_\alpha) = \text{Support}(\psi_\alpha)$ is locally finite, and
(3) $\sum_{\alpha \in J} \varphi_\alpha(x) = \sum_{\alpha \in J} \psi_\alpha(x)/\Psi_\alpha(x) = 1$.

That is, $\{\psi_\alpha\}_{\alpha \in J}$ is (by definition) a partition of unity dominated by $\{U_\alpha\}_{\alpha \in J}$. □

Theorem 41.8

Theorem 41.8. Let X be a paracompact Hausdorff space. Let $\mathcal{C}$ be a collection of subsets of X and for each $X \in \mathcal{C}$ let $\varepsilon_C > 0$. If $\mathcal{C}$ is locally finite, then there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) > 0$ for all x, and $f(x) \leq \varepsilon_C$ for $x \in C$.

Proof. Since $\mathcal{C}$ is locally finite then (by definition) for each $x \in X$ there is a neighborhood of x which intersects only finitely many elements of $\mathcal{C}$, so create an open covering of X with such neighborhoods and denote it $\{U_\alpha\}_{\alpha \in J}$. By Theorem 41.7, there is a partition of unity $\{\varphi_\alpha\}_{\alpha \in J}$ on X dominated by $\{U_\alpha\}_{\alpha \in J}$.

Theorem 41.8

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Proof. Since $\mathcal{C}$ is locally finite then (by definition) for each $x \in X$ there is a neighborhood of x which intersects only finitely many elements of $\mathcal{C}$, so create an open covering of X with such neighborhoods and denote it $\{U_\alpha\}_{\alpha \in J}$. By Theorem 41.7, there is a partition of unity $\{\varphi_\alpha\}_{\alpha \in J}$ on X dominated by $\{U_\alpha\}_{\alpha \in J}$.

For a given $\alpha \in J$, let δ_α be the minimum of the $\varepsilon_C > 0$ as C ranges over the elements of $\mathcal{C}$ which intersect the support of φ_α (by definition of "partition of unity," Support$(\varphi_\alpha) \subset U_\alpha$ and by construction U_α intersects only finitely many $C \in \mathcal{C}$, so there are finitely many such C). If there are so such $C \in \mathcal{C}$, then set $\delta_\alpha = 1$.

Theorem 41.8

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Proof. Since $\mathcal{C}$ is locally finite then (by definition) for each $x \in X$ there is a neighborhood of x which intersects only finitely many elements of $\mathcal{C}$, so create an open covering of X with such neighborhoods and denote it $\{U_\alpha\}_{\alpha \in J}$. By Theorem 41.7, there is a partition of unity $\{\varphi_\alpha\}_{\alpha \in J}$ on X dominated by $\{U_\alpha\}_{\alpha \in J}$.

For a given $\alpha \in J$, let δ_α be the minimum of the $\varepsilon_C > 0$ as C ranges over the elements of $\mathcal{C}$ which intersect the support of φ_α (by definition of "partition of unity," Support$(\varphi_\alpha) \subset U_\alpha$ and by construction U_α intersects only finitely many $C \in \mathcal{C}$, so there are finitely many such C). If there are so such $C \in \mathcal{C}$, then set $\delta_\alpha = 1$.

Theorem 41.8 (continued)

Proof (continued). Define $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ (for given $x \in X$, this is nonzero for only finitely many $\alpha \in J$). Since $\varphi_\alpha(x) : X \to [0, 1]$, $\varphi_\alpha(x) > 0$ for some $\alpha \in J$ and for such α we have $\delta_\alpha > 0$, then f is positive valued for all $x \in X$, as claimed. If $x \notin \text{Support}(\varphi_\alpha)$ then $\varphi_\alpha(x) = 0$; if $x \in \text{Support}(\varphi_\alpha)$ and $x \in C$ then $\delta_\alpha \le \varepsilon_C$. So for any $x \in C$ we have $\delta_\alpha \varphi_\alpha(x) \le \varepsilon_C \varphi_\alpha(x) \le \varepsilon_C$ for arbitrary $\alpha \in J$ and since, $\sum_{\alpha \in J} \varphi_\alpha(x) = 1$,

$$f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha \le \sum_{\alpha \in J} \varepsilon_C \varphi_\alpha(x) = \varepsilon_C \sum_{\alpha \in J} \varphi_\alpha = \varepsilon_C,$$

as claimed.

Theorem 41.8 (continued)

Proof (continued). Define $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ (for given $x \in X$, this is nonzero for only finitely many $\alpha \in J$). Since $\varphi_\alpha(x) : X \to [0,1]$, $\varphi_\alpha(x) > 0$ for some $\alpha \in J$ and for such α we have $\delta_\alpha > 0$, then f is positive valued for all $x \in X$, as claimed. If $x \notin \text{Support}(\varphi_\alpha)$ then $\varphi_\alpha(x) = 0$; if $x \in \text{Support}(\varphi_\alpha)$ and $x \in C$ then $\delta_\alpha \le \varepsilon_C$. So for any $x \in C$ we have $\delta_\alpha \varphi_\alpha(x) \le \varepsilon_C \varphi_\alpha(x) \le \varepsilon_C$ for arbitrary $\alpha \in J$ and since, $\sum_{\alpha \in J} \varphi_\alpha(x) = 1$,

$$f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha \le \sum_{\alpha \in J} \varepsilon_C \varphi_\alpha(x) = \varepsilon_C \sum_{\alpha \in J} \varphi_\alpha = \varepsilon_C,$$

as claimed.

Finally, $\{\text{Support}(\varphi_\alpha)\}_{\alpha \in J}$ is locally finite (by the definition of "partition of unity") so for any $x \in X$, there is a neighborhood W of x such that W intersects only finitely many $\text{Support}(\varphi_\alpha)$'s. So on W, $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ is the sum of finitely many continuous (real valued) functions and so f is continuous on W; that is, f restricted to each such W is continuous.

Theorem 41.8 (continued)

Proof (continued). Define $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ (for given $x \in X$, this is nonzero for only finitely many $\alpha \in J$). Since $\varphi_\alpha(x) : X \to [0,1]$, $\varphi_\alpha(x) > 0$ for some $\alpha \in J$ and for such α we have $\delta_\alpha > 0$, then f is positive valued for all $x \in X$, as claimed. If $x \notin \text{Support}(\varphi_\alpha)$ then $\varphi_\alpha(x) = 0$; if $x \in \text{Support}(\varphi_\alpha)$ and $x \in C$ then $\delta_\alpha \leq \varepsilon_C$. So for any $x \in C$ we have $\delta_\alpha \varphi_\alpha(x) \leq \varepsilon_C \varphi_\alpha(x) \leq \varepsilon_C$ for arbitrary $\alpha \in J$ and since, $\sum_{\alpha \in J} \varphi_\alpha(x) = 1$,

$$f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha \leq \sum_{\alpha \in J} \varepsilon_C \varphi_\alpha(x) = \varepsilon_C \sum_{\alpha \in J} \varphi_\alpha = \varepsilon_C,$$

as claimed.

Finally, $\{\text{Support}(\varphi_\alpha)\}_{\alpha \in J}$ is locally finite (by the definition of "partition of unity") so for any $x \in X$, there is a neighborhood W of x such that W intersects only finitely many $\text{Support}(\varphi_\alpha)$'s. So on W, $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ is the sum of finitely many continuous (real valued) functions and so f is continuous on W; that is, f restricted to each such W is continuous. By by Theorem 18.2(f), f is continuous on X, as claimed. □

Theorem 41.8 (continued)

Proof (continued). Define $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ (for given $x \in X$, this is nonzero for only finitely many $\alpha \in J$). Since $\varphi_\alpha(x) : X \to [0,1]$, $\varphi_\alpha(x) > 0$ for some $\alpha \in J$ and for such α we have $\delta_\alpha > 0$, then f is positive valued for all $x \in X$, as claimed. If $x \not\in \text{Support}(\varphi_\alpha)$ then $\varphi_\alpha(x) = 0$; if $x \in \text{Support}(\varphi_\alpha)$ and $x \in C$ then $\delta_\alpha \le \varepsilon_C$. So for any $x \in C$ we have $\delta_\alpha \varphi_\alpha(x) \le \varepsilon_C \varphi_\alpha(x) \le \varepsilon_C$ for arbitrary $\alpha \in J$ and since, $\sum_{\alpha \in J} \varphi_\alpha(x) = 1$,

$$f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha \le \sum_{\alpha \in J} \varepsilon_C \varphi_\alpha(x) = \varepsilon_C \sum_{\alpha \in J} \varphi_\alpha = \varepsilon_C,$$

as claimed.

Finally, $\{\text{Support}(\varphi_\alpha)\}_{\alpha \in J}$ is locally finite (by the definition of "partition of unity") so for any $x \in X$, there is a neighborhood W of x such that W intersects only finitely many Support(φ_α)'s. So on W, $f(x) = \sum_{\alpha \in J} \delta_\alpha \varphi_\alpha(x)$ is the sum of finitely many continuous (real valued) functions and so f is continuous on W; that is, f restricted to each such W is continuous. By by Theorem 18.2(f), f is continuous on X, as claimed. □