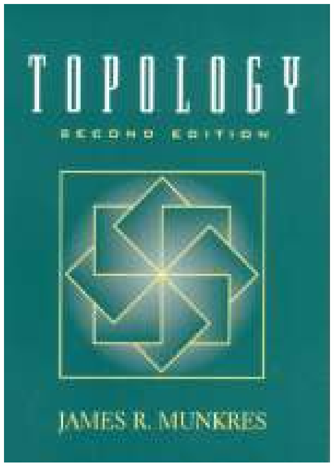
Introduction to Topology

Chapter 6. Metrization Theorems and Paracompactness

Section 42. The Smirnov Metrization Theorem—Proofs of Theorems



Theorem 42.1

Theorem 42.1. The Smirnov Metrization Theorem.
A topological space X is metrizable if and only if it is a paracompact Hausdorff space that is locally metrizable.

Proof. Suppose that X is metrizable. Then X is locally metrizable. So, by Theorem 41.4, X is paracompact. Every metric space is Hausdorff (see page 129), and one of the implications follows.

Conversely, suppose that X is a paracompact Hausdorff space that is locally metrizable. Since X is locally metrizable, there is (by definition) an open covering of X by open sets that are metrizable. Since X is paracompact and Hausdorff then, by Theorem 41.1, X is normal. Since X is paracompact then (by definition of paracompact) there is a locally finite open refinement $\mathcal{C}$ of this covering that covers X. So each C of $\mathcal{C}$ is metrizable since $\mathcal{C}$ is a refinement of the covering by metrizable open sets. Let $d_C : C \times C \to \mathbb{R}$ be a metric that gives the topology of C. For $x \in C$, let $B_C(x,\varepsilon) = \{y \in C \mid d_C(x,y) < \varepsilon\}$. Since $B_C(x,\varepsilon)$ is open in C (under the subspace topology) and $B_C(x,\varepsilon) \subset C$, then $B_C(x,\varepsilon)$ is open in X.

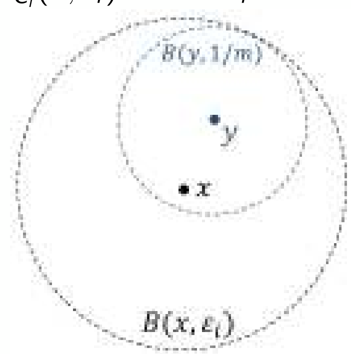
Theorem 42.1 (continued 1)

Proof (continued). For $m \in \mathbb{N}$, let $\mathcal{A}_m$ be the covering of X by all of these open balls of readius $1/m$: $\mathcal{A}_m = \{B_C(x,1/m) \mid x \in C \text{ and } C \in \mathcal{C}\}$. Since X is paracompact then (by definition) there is a locally finite open refinement $\mathcal{D}_m$ of $\mathcal{A}_m$ that covers X. Let $\mathcal{D}$ be the union of the collection $\mathcal{D}_m$: $\mathcal{D} = \cup_{m\in\mathbb{N}}\mathcal{D}_m$. Then $\mathcal{D}$ is, by definition, countably locally finite (that is, $\mathcal{D}$ is a countable union [over $m \in \mathbb{N}$] of locally finite collections [the $\mathcal{D}_m$'s]). We now show that $\mathcal{D}$ is a basis for X.

Let $x \in X$ and let U be a neighborhood of x. Now x belongs to only finitely many elements of $\mathcal{C}$ (since $\mathcal{C}$ is locally finite), say $C_1, C_2, \ldots, C_k$. Then $U \cap C_i$ is a neighborhood of x (since $\mathcal{C}$ is an open covering and so each C_i is open) in the set C_i, so there is $\varepsilon_i > 0$ such that $B_C(x,\varepsilon) \subset U \cap C_i$ (since C_i has the metric topology induced by d_{C_i}). Let $m \in \mathbb{N}$ such that $2/m < \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k\}$. Since $\mathcal{D}_m$ covers X then there is $D \in \mathcal{D}_m$ with $x \in \mathcal{D}_m$.

Theorem 42.1 (continued 2)

Proof (continued). Since $\mathcal{D}_m$ is a refinement of $\mathcal{A}_m$ and $\mathcal{A}_m$ includes all $B_C(x,1/m)$ for $z \in C$, $C \in \mathcal{C}$, then there must be some $y \in C$ where $C \in \mathcal{C}$ such that $B_C(y,1/m) \in \mathcal{A}_m$ and $D \subset B_C(y,1/m)$. Since $x \in D \subset B_C(y,1/m) \subset C$, then $x \in C$ and so $C = C_i$ for some $i \in \{1,2,\ldots,k\}$. Since $B_C(y,1/m)$ has diameter at most $2/m < \varepsilon_i$, then $x \in D \subset B_{C_i}(y,1/m) \subset B_{C_i}(x,\varepsilon_i) \subset U \cap C_i \subset U$:



Theorem 42.1 (continued 3)

Theorem 42.1. The Smirnov Metrization Theorem.
A topological space X is metrizable if and only if it is a paracompact Hausdorff space that is locally metrizable.

Proof. Since $x \in X$ and neighborhood U of x are arbitrary and $D \in \mathcal{D}_m \subset \mathcal{D}$ with $D \subset U$, then $\mathcal{D}$ is a basis for the topology on X and $\mathcal{D}$ is countable locally finite as explained above. Since X is normal (as shown above) then X is regular and so the the Nagata-Smirnov Metrization Theorem (Theorem 40.3), X is metrizable. □