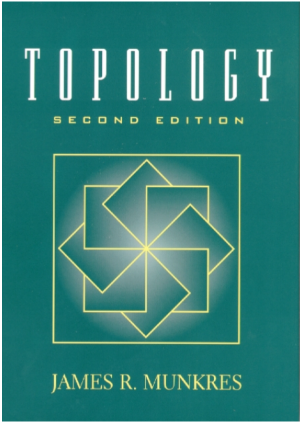
Introduction to Topology

Chapter 6. Metrization Theorems and Paracompactness

Section 42. The Smirnov Metrization Theorem—Proofs of Theorems



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Proof (continued). For $m \in \mathbb{N}$, let $\mathcal{A}_m$ be the covering of X by all of these open balls of readius $1/m$: $\mathcal{A}_m = \{B_C(x, 1/m) \mid x \in C \text{ and } C \in \mathcal{C}\}$. Since X is paracompact then (by definition) there is a locally finite open refinement $\mathcal{D}_m$ of $\mathcal{A}_m$ that covers X. Let $\mathcal{D}$ be the union of the collection $\mathcal{D}_m$: $\mathcal{D} = \cup_{m \in \mathbb{N}} \mathcal{D}_m$. Then $\mathcal{D}$ is, by definition, countably locally finite (that is, $\mathcal{D}$ is a countable union [over $m \in \mathbb{N}$] of locally finite collections [the $\mathcal{D}_m$'s]). We now show that $\mathcal{D}$ is a basis for X.

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Let $x \in X$ and let U be a neighborhood of x. Now x belongs to only finitely many elements of $\mathcal{C}$ (since $\mathcal{C}$ is locally finite), say $C_1, C_2, \ldots, C_k$. Then $U \cap C_i$ is a neighborhood of x (since $\mathcal{C}$ is an open covering and so each C_i is open) in the set C_i, so there is $\varepsilon_i > 0$ such that $B_C(x, \varepsilon) \subset U \cap C_i$ (since C_i has the metric topology induced by d_{C_i}).

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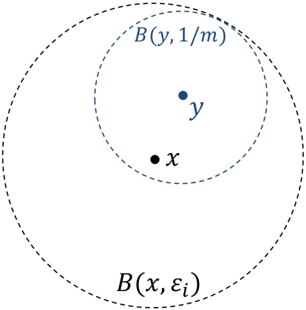

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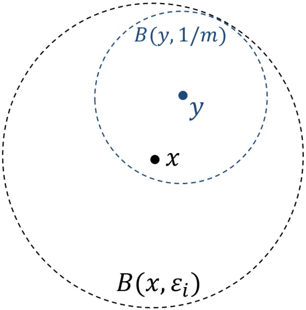

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