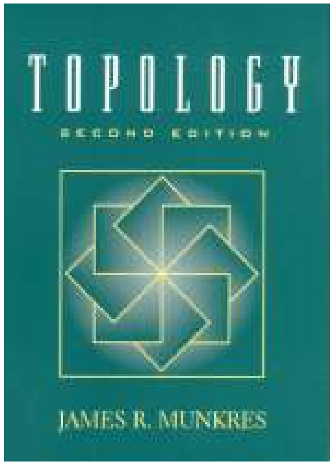
Introduction to Topology

Chapter 7. Complete Metric Spaces and Function Spaces
Section 43. Complete Metric Spaces—Proofs of Theorems



Lemma 43.1

Lemma 43.1. A metric space (X, d) is complete if every Cauchy sequence in X has a convergent subsequence.

Proof. Let (x_n) be a Cauchy sequence in (X, d). Let (x_{n_i}) be a subsequence of (x_n) that converges to some $x \in X$. Let $\varepsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for all $n, m \geq N_1$. Since $(x_{n_i}) \to x$, let $N_2 \in \mathbb{N}$ such that for $n_i \geq N_2$ we have $d(x_{n_i}, x) < \varepsilon$. So with $n \geq N_1$ and $n_i \geq N_2$ we have

$$d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So arbitrary Cauchy sequence (x_n) converges (to x) and (X, d) is complete. □

Theorem 43.2

Theorem 43.2. Euclidean space $\mathbb{R}^k$ (where $k \in \mathbb{N}$) is complete in either of its usual metrics, the Euclidean metric d or the square metric ρ.

Proof. Let (x_n) be a Cauchy sequence in $(\mathbb{R}^k, \rho)$. Notice that with $\varepsilon = 1$ there is $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $\rho(x_n, x_m) \leq 1$, so

$$M = \max\{\rho(x_1, \mathbf{0}), \rho(x_2, \mathbf{0}), \ldots, \rho(x_{N-1}), \rho(x_N, \mathbf{0}) + 1\}$$

is an upper bound for $\rho(x_n, \mathbf{0})$ for all $n \in \mathbb{N}$ (that is, (x_n) is a bounded sequence). So $(x_n) \subset [-M, M]^k$. Now the cube $[-M, M]^k$ is closed and bounded and so is compact by Theorem 27.3 (The Heine-Borel Theorem) in both $(\mathbb{R}^k, \rho)$ and $(\mathbb{R}^k, d)$. By Theorem 28.2, $[-M, M]^k$ is sequentially compact and so (by the definition of sequentially compact) (x_n) has a convergent subsequence; so by Lemma 43.1, $(\mathbb{R}^k, \rho)$ is complete.
By Theorem 20.3, ρ and d induce the same topology on $\mathbb{R}^k$ (namely, the product topology) so a sequence is Cauchy (or convergent) relative to ρ if and only if it is Cauchy (or convergent, respectively) relative to d. So the same argument given above shows that $(\mathbb{R}^k, d)$ is also complete. □

Lemma 43.3

Lemma 43.3. Let X be the product space $X = \prod_{\alpha \in J} X_\alpha$ (under the product topology) and let $(\mathbf{x}_n)$ be a sequence of points in X. Then $\mathbf{x}_n \to \mathbf{x}$ if and only if $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for all $\alpha \in J$.
Proof. Suppose $\mathbf{x}_n \to \mathbf{x}$. Each projection π_α is continuous (see the proof of Theorem 19.6), so fall all $\alpha \in J$,

$$\begin{aligned} \lim_{n\to\infty} \pi_\alpha(\mathbf{x}_n) &= \pi_\alpha\left(\lim_{n\to\infty} \mathbf{x}_n\right) \text{ by Theorem 21.3} \\ &= \pi_\alpha(\mathbf{x}). \end{aligned}$$

Suppose $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for all $\alpha \in J$. Let $U = \prod_{\alpha \in J} U_\alpha$ be a basis element for X in the product topology which contains $\mathbf{x}$ (so by Theorem 19.1, $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$). For each $\alpha \in J$ for which $U_\alpha \neq X_\alpha$, choose $N_\alpha \in \mathbb{N}$ such that $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$ for all $n \geq N_\alpha$ (such $N_\alpha \in \mathbb{N}$ exists since $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$). Let N be the largest of the N_α (since there are only finitely many N_α, there is a largest). Then for all $n \geq N$, $\mathbf{x}_n \in U$. Therefore, since U is an arbitrary basis element, then $\mathbf{x}_n \to \mathbf{x}$. □

Theorem 43.4

Theorem 43.4. There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is complete.

Proof. Consider $D(\mathbf{x}, \mathbf{y} = \sup_{i \in \mathbb{N}}\{\overline{d}(x_i, y_i)/i\}$ where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. Then D is a metric on $\mathbb{R}^\omega$ (see Section 20) and D induces the product topology by Theorem 20.5. Now let $(\mathbf{x}_n)$ be a Cauchy sequence i $(\mathbb{R}^\omega, D)$. For all $i \in \mathbb{N}$,

$$\overline{d}(x_i, y_i/i = \overline{d}(\pi_i(\mathbf{x}), \pi_i(\mathbf{y}))/i \leq \sup_{i \in \mathbb{N}}\{\overline{d}(x_i, y_i)/i\} = D(\mathbf{x}, \mathbf{y}),$$

so for all $i \in \mathbb{N}$ we have $\overline{d}(\pi_i(\mathbf{x}, \pi_i(\mathbf{y})) \leq iD(\mathbf{x}, \mathbf{y})$. So for a fixed i, since $(\mathbf{x}_n)$ is Cauchy, we have for all $\varepsilon > 0$ that there exists $N \in \mathbb{N}$ such that for $m, n \geq N$ we have $\overline{d}(\pi_i(\mathbf{x}_n), \pi_i(\mathbf{x}_m)) \leq iD(\mathbf{x}_n, \mathbf{x}_m) < i(\varepsilon/i) = \varepsilon$, and so $(\pi_i(\mathbf{x}_n))$ is a Cauchy sequence in $\mathbb{R}$. So $\pi_i(\mathbf{x}_n) \to a_i$. Consider $\mathbf{a} = (a_1, a_2, \ldots) \in \mathbb{R}^\omega$. Then by Lemma 43.3, $\mathbf{x}_n \to \mathbf{a}$ and so $(\mathbb{R}^\omega, D)$ is complete. □

Theorem 43.5

Theorem 43.5. If the space Y is complete in the metric d, then the space Y^J is complete in the uniform metric $\overline{\rho}$ corresponding to d.

Proof. If (Y, d) is complete then $(Y, \overline{d})$ is complete (see the Note in the class notes before Lemma 43.1). Let (f_n) be a Cauchy sequence in $(Y, \overline{\rho})$. For any $\alpha \in J$ we have

$$\overline{d}(f_x(\alpha), f_m(\alpha)) \leq \sup\{\overline{d}(f_n(\alpha), f_m(\alpha)) \mid \alpha \in J\} = \overline{\rho}(f_n, f_m), \qquad (*)$$

so $(f_n(\alpha))$ is a Cauchy sequence in $(y, \overline{d})$. Since $(Y, \overline{d})$ is complete, then there is $y_\alpha \in Y$ such that $f_n(\alpha) \to y_\alpha$ with respect to $\overline{d}$. Define $f : J \to Y$ as $f(\alpha) = y_\alpha$; so $f \in Y^J$. We next show that $f_n \to f$ with respect to $\overline{\rho}$.

Theorem 43.5 (continued)

Theorem 43.5. If the space Y is complete in the metric d, then the space Y^J is complete in the uniform metric $\overline{\rho}$ corresponding to d.

Proof (continued). Let $\varepsilon > 0$. There is $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$ we have $\overline{\rho}(f_n, f_m) < \varepsilon/2$ since (f_n) is Cauchy with respect to $\overline{\rho}$. So by $(*)$, for all $\alpha \in J$, $\overline{d}(f_n(\alpha), f_m(\alpha)) < \varepsilon/4$ whenever $m, n \geq N_1$. Since $f_m(\alpha) \to y_\alpha = f(\alpha)$ with respect to $\overline{d}$ then there is $N_2 \in \mathbb{N}$ such that for all $m \geq N_2$ we have $\overline{d}(f_m(\alpha), f(\alpha)) < \varepsilon/4$. So for given $\alpha \in J$, with $n \geq N_1$ and $m \geq N_2$ we have

$$\overline{d}(f_n(\alpha), f(\alpha)) \leq \overline{d}(f_n(\alpha), f_m(\alpha)) + \overline{d}(f_m(\alpha), f(\alpha)) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

That is, for all $\alpha \in J$, if $n \geq N_1$ then $\overline{d}(f_n(\alpha), f(\alpha)) < \varepsilon/2$. Hence, for all $n \geq N_1$ we have $\overline{\rho}(f_n, f) = \sup\{\overline{d}(f_n(\alpha), f(\alpha)) \mid \alpha \in J\} \leq \varepsilon/2$, and so $(f_n) \to f$ with respect to $\overline{\rho}$. Since (f_n) is an arbitrary Cauchy sequence in $(Y^J, \overline{\rho})$, the $(Y^J, \overline{\rho})$ is complete. □

Theorem 43.6

Theorem 43.6. Let X be a topological space and let (Y, d) be a metric space. The set $\mathcal{C}(X, Y)$ of continuous functions is closed in Y^X under the uniform metric. So is the set $\mathcal{B}(X, Y)$ of bounded functions. Therefore, if Y is a complete metric space, then both $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ are complete metric spaces under the uniform metric.

Proof. Let $(f_n) \to f$ in Y^X relative to $\overline{\rho}$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\overline{\rho}(f_n, f) < \varepsilon$. So for all $x \in X$ and for all $n \geq N$ we have

$$\overline{d}(f_n(x), f(x)) \leq \sup\{\overline{d}(f_n(x), f(x)) \mid x \in X\} = \overline{\rho}(f_n, f) < \varepsilon.$$

Therefore (f_n) converges uniformly to f.

Now we show that $\mathcal{C}(X, Y)$ is closed in Y^X relative to $\overline{\rho}$. Let $f \in Y^X$ where f is a limit point of $\mathcal{C}(X, Y)$. By Theorem 17.6, f is a point of the closure of $\mathcal{C}(X, Y)$ and so by The Sequence Lemma (Lemma 21.2) there is a sequence (f_n) of elements of $\mathcal{C}(X, Y)$ which converges to f relative to $\overline{\rho}$.

Theorem 43.6 (continued 1)

Proof (continued). But as shown above, this means that f is the uniform limit of (f_n) and so by the Uniform Limit Theorem (Theorem 21.6) f is continuous. That is, $f \in \mathcal{C}(X,Y)$ and since f is an arbitrary limit point of $\mathcal{C}(X,Y)$, then $\mathcal{C}(X,Y)$ contains all of its limit points and so is closed by Theorem 17.6, as claimed.

Now to show that $\mathcal{B}(X,Y)$ is closed in Y^X relative to $\overline{\rho}$. Let f be a limit point of $\mathcal{B}(X,Y)$. As above, there is a sequence (f_n) of elements of $\mathcal{B}(X,Y)$ where $(f_n) \to f$ relative to $\overline{\rho}$. So there exists $N \in \mathbb{N}$ such that $\overline{\rho}(f_N, f) < 1/2$. Then for all $x \in X$ we have

$$\overline{d}(f_N(x), f(x) \le \sup\{\overline{d}(f_N(x), f(x) \mid x \in X\} = \overline{\rho}(f_N, f) < 1/2.$$

Since $\overline{d}(f_N(x), f(x)) = \min\{d(f_N(x), f(x)), 1\}$ this means that for all $x \in X$ we have $d(f_N(x), f(x)) < 1/2$. Let M be the diameter of the set $f_N(x)$ (which exists as a finite number since f_N is bounded) then by the Triangle Inequality for d, the diameter of $f(X)$ is at most $M+1$. Hence $f \in \mathcal{B}(X,Y)$ and, as above, $\mathcal{B}(X,Y)$ is closed, as claimed.

Theorem 43.6 (continued 2)

Theorem 43.6. Let X be a topological space and let (Y,d) be a metric space. The set $\mathcal{C}(X,Y)$ of continuous functions is closed in Y^X under the uniform metric. So is the set $\mathcal{B}(X,Y)$ of bounded functions. Therefore, if Y is a complete metric space, then both $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ are complete metric spaces under the uniform metric.

Proof (continued). Suppose (Y,d) is complete. By Theorem 43.5, we know that if Y is complete in the metric d then Y^X is complete in the uniform metric $\overline{\rho}$. So if (f_n) is any Cauchy sequence in either $\mathcal{C}(X,Y)$ or $\mathcal{B}(X,Y)$, then (f_n) is a Cauchy sequence in Y^X (since $\mathcal{C}(X,Y) \subset Y^X$ and $\mathcal{B}(X,Y) \subset Y^X$) and since Y^X is complete then $(f_n) \to f$ for some $f \in Y^X$. Then f is a limit point of both $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ by The Sequence Lemma (Lemma 21.2), and as shown above, $f \in \mathcal{C}(X,Y)$ and $f \in \mathcal{B}(X,Y)$. Therefore, if (Y,d) is complete then both $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ are complete with respect to $\overline{\rho}$. □

Theorem 43.7

Theorem 43.7. Let (X,d) be a metric space. There is an isometric embedding of X into a complete space.

Proof. Let $\mathcal{B}(X,\mathbb{R})$ be the set of all bounded functions mapping X into $\mathbb{R}$. Let $x_0 \in X$ be fixed. Given $a \in X$, define $\varphi_a : X \to \mathbb{R}$ as $\varphi_a(x) = d(x,a) - d(x,x_0)$. For any $b \in X$ we have by the Triangle Inequality for d that for all $x \in X$ we have

$$d(x,a) \le d(x,b) + d(a,b) \text{ and } d(x,b) \le d(x,a) + d(a,b)$$

and combining these inequalities gives $-d(a,b) \le d(x,a) - d(x,b) \le d(a,b)$ or

$$|d(x,a) - d(x,b)| \le d(a,b). \qquad (*)$$

With $b = x_0$, we then have $|\varphi_1(x)| \le d(a,x_0)$ for all $x \in X$. Therefore, φ_1 is bounded and $\varphi_a \in \mathcal{B}(X,\mathbb{R})$.

Theorem 43.7 (continued 1)

Proof (continued). Define $\Phi : X \to \mathcal{B}(X,\mathbb{R})$ as $\Phi(a) = \varphi_a$. We now show that Φ is an isometric embedding of (X,d) into the complete metric space $(\mathcal{B}(X,\mathbb{R}),\rho)$. Notice that since $(\mathbb{R}, |\cdot|)$ is complete, then $(\mathcal{B}(X,\mathbb{R}),\overline{\rho})$ is complete by Theorem 43.6. Since $\overline{\rho}(f,g) = \min\{\rho(f,g),1\}$ (see the Note before the statement of this theorem) then completeness of $(\mathcal{B}(X,\mathbb{R}),\rho)$ is equivalent to the completeness of $(\mathcal{B}(X,\mathbb{R}),\overline{\rho})$ (see the Note before the statement of Lemma 43.1 in these class notes, or see page 264). So $(\mathcal{B}(X,\mathbb{R}),\rho)$ is in fact a complete metric space. Next, for the isometry claim, let $a,b \in X$. Then

$$\begin{aligned}
\rho(\varphi_a,\varphi_b) &= \sup\{|\varphi_a(x) - \varphi_b(x)| \mid x \in X\} \text{ since the metric on } \mathbb{R} \text{ is } |\cdot| \\
&= \sup\{|(d(x,a) - d(x,x_0)) - (d(x,b) - d(x,x_0))| \mid x \in X\} \\
&\quad \text{ by the definition of } \varphi_a \text{ and } \varphi_b \\
&= \sup\{|d(x,a) - d(x,b)| \mid x \in X\} \\
&\le d(a,b) \text{ by } (*).
\end{aligned}$$

Theorem 43.7 (continued 2)

Theorem 43.7. Let (X, d) be a metric space. There is an isometric embedding of X into a complete space.

Proof (continued). So $\rho(\varphi_a, \varphi_b) \le d(a, b)$. But when $x = a$ $|d(x, a) - d(x, b)| = d(a, b)$ and so $\rho(\varphi_a, \varphi_b) = \sup\{|d(x, a) - d(x, b)| \mid x \in X\} \ge d(a, b)$. Therefore, $\rho(\varphi_a, \varphi_b) = d(a, b)$ and the mapping $\Phi : X \to \mathcal{B}(X, \mathbb{R})$ is an isometry (that is, $d(a, b) = \rho(\varphi_a, \varphi_b) = \rho(\Phi(a), \Phi(b))$). □