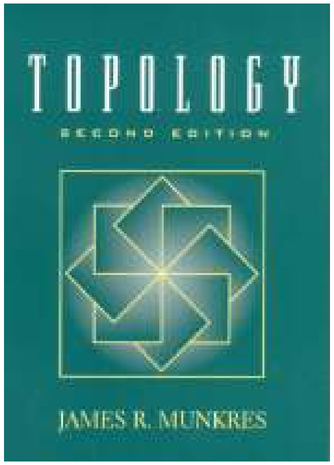
# Introduction to Topology

**Chapter 7. Complete Metric Spaces and Function Spaces**
Section 45. Compactness in Metric Spaces—Proofs of Theorems








## Theorem 45.1

**Theorem 45.1.** A metric space $(X, d)$ is compact if and only if it is complete and totally bounded.

**Proof.** If $X$ is a compact metric space, then $X$ is complete as argued in the note above. Any covering of $X$ by $\varepsilon$-balls has a finite subcover by the compactness of $X$, and so $X$ is totally bounded.

Conversely, let $X$ be complete and totally bounded. Let $(x_n)$ be a sequence in $X$. Cover $X$ by finitely many $\varepsilon = 1$ balls using the total boundedness of $X$. At least one of these balls, say $B_1$, contains $x_n$ for infinitely many values of $n \in \mathbb{N}$. Let $J_1 \in \mathbb{N}$ consist of precisely these indices. Next, cover $X$ by finitely may $\varepsilon = 1/2$ balls. Since $J_1$ is infinite, at least one of these balls, say $B_2$, must contain $x_n$ for infinitely many values of $n \in J_1$. Let $J_2 \subset J_1 \subset \mathbb{N}$ consists of precisely these indices.





## Theorem 45.1 (continued)

**Theorem 45.1.** A metric space $(X, d)$ is compact if and only if it is complete and totally bounded.

**Proof.** Inductively create ball $B_n$ of radius $\varepsilon = 1/n$ and set $J_n \subset J_{n-1} \subset \mathbb{N}$ of indices of $x_i \in B_n$. Now choose $n_1 \in J_1$ and inductively choose $n_k \in J_k$ for $k \geq 2$ such that $n_k > n_{k-1}$. Now for $i, j \geq k$, the indices $n_i$ and $n_j$ both belong to $J_k$. Therefore, for all $i, j \geq k$, then points $x_{n_i}$ and $x_{n_j}$ are contained in ball $B_k$ of radius $1/k$. Hence, $(x_{n_i})$ is a Cauchy sequence. Since $X$ is complete, then $(x_{n_i})$ converges. Since $(x_n)$ is an arbitrary sequence in $X$ and $(x_{n_i}) \subset (x_n)$ is a convergent subsequence, then $X$ is sequentially compact. So, by Theorem 28.2, $X$ is compact. □





## Lemma 45.2

**Lemma 45.2.** Let $X$ be a topological space and let $(Y, d)$ be a metric space. If the subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is totally bounded under the uniform metric corresponding to $d$, then $\mathcal{F}$ is equicontinuous under $d$.

**Proof.** Suppose $\mathcal{F}$ is totally bounded under the uniform metric

$$\overline{\rho}(f, g) = \sup\{\overline{d}(f(x), g(x)) \mid x \in X\}$$

where

$$\overline{d}(f(x), g(x)) = \min\{d(f(x), g(x)), 1\}.$$

Let $\varepsilon > 0$, where $\varepsilon < 1$, and let $x_0 \in X$.

Set $\delta = \varepsilon/3$. By the total boundedness of $\mathcal{F}$, there is a finite $\delta$-ball covering of $\mathcal{F}$, say $B(f_1, \delta), B(f_2, \delta), \ldots, B(f_n, \delta)$. Since each $f_i$ is continuous, there is a neighborhood $U$ of $x_0$ such that for $i = 1, 2, \ldots, n$ we have $d(f_i(x), f_i(x_0)) < \delta$ for all $x \in U$ (choose such an open neighborhood $U_i$ of $x_0$ for each $i = 1, 2, \ldots, n$ and let $U = \cap_{i=1}^n U_i$).

## Lemma 45.2 (continued)

**Proof (continued).** Let $f \in \mathcal{F}$. Then $f$ belongs to some $\delta$-ball, say $B(f_i, \delta) = \{g \in \mathcal{C}(X,Y) \mid \overline{\rho}(f_i, g) < \delta\}$. Then for all $x \in U$ we have

(1) $\overline{d}(f(x), f_i(x)) = \min\{d(f(x), f_i(x)), 1\} < \delta$ since $f \in B(f_i, \delta)$ and so $\overline{\rho}(f, f_i) < \delta$,

(2) $d(f_i(x), f_i(x_0)) < \delta$ since $x \in U$,

(3) $\overline{d}(f_i(x_0), f(x_0)) = \min\{d(f_i(x_0), f(x_0)), 1\} < \delta$ since $f \in \mathcal{B}(f_i, \delta)$ and so $\overline{\rho}(f, f_i) < \delta$.

Since $\delta < 1$ (actually, $\delta = \varepsilon/3 < 1/3$), we have from (1) and (3) that $d(f(x), f_i(x)) < \delta$ and $d(f_i(x_0), f(x_0)) < \delta$. Therefore

$$\begin{aligned} d(f(x), f(x_0)) &\le d(f(x), f_i(x)) + d(f_i(x), f_i(x_0)) + d(f_i(x_0), f(x_0)) \\ &\qquad \text{by the Triangle Inequality} \\ &< \delta + \delta + \delta = \varepsilon. \end{aligned}$$

Therefore $\mathcal{F}$ is equicontinuous at $x_0$ and since $x_0$ is an arbitrary point of $X$ then set $\mathcal{F}$ is equicontinuous. □





## Lemma 45.3

**Lemma 45.3.** Let $X$ be a topological space and let $(Y, d)$ be a metric space. Assume $X$ and $Y$ are compact. If the subset $\mathcal{F}$ of $\mathcal{C}(X,Y)$ is equicontinuous under $d$, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to $d$.

**Proof.** Recall that the sup metric is

$$\rho(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}$$

(see Section 43). Since $X$ is compact and the elements of $\mathcal{C}(X,Y)$ are continuous, then $\rho$ is defined on $\mathcal{C}(X,Y)$. Total boundedness under $\rho$ is equivalent to total boundedness under the uniform metric

$$\overline{\rho}(f,g) = \sup\{\overline{d}(f(x), g(x)) \mid x \in X\}$$

where

$$\overline{d}(f(x), g(x)) = \min\{d(f(x), g(x)), 1\}$$

since when $\varepsilon < 1$, every $\varepsilon$-ball under $\rho$ is also an $\varepsilon$-ball under $\underline{\rho}$ and conversely. So without loss of generality we assume the metric is $\rho$.





## Lemma 45.3 (continued 1)

**Proof (continued).** Let $\mathcal{F} \subset \mathcal{C}(X,Y)$ be equicontinuous. Let $\varepsilon > 0$. Set $\delta = \varepsilon/3$. By the equicontinuity, for any $a \in X$, there is a corresponding neighborhood $U_a$ of $a$ such that $d(f(x), f(a)) < \delta$ for all $x \in U_a$ and for all $f \in \mathcal{F}$. Since $X$ is compact, the open covering by all such $U_z$ has a finite subcover $U_{a_1}, U_{a_2}, \ldots, U_{a_k}$. Since $Y$ is compact then there is a finite cover of $Y$ by open sets $V_1, V_2, \ldots, V_m$ of diameter less than $\delta$.

Let $J$ be the collection of all functions $\alpha : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, m\}$. Given $\alpha \in J$, if there exists $f \in \mathcal{F}$ such that $f(a_i) \in V_{a(i)}$ for each $i = 1, 2, \ldots, k$, then choose one such function and denote it as $f_\alpha$ (since the $V_i$ cover $Y$ and we are considering all such $\alpha$, then for each $f \in \mathcal{F}$ such an $\alpha$ exists; since we choose at most one $f \in \mathcal{F}$ to be associated with each $\alpha$ then the collection of $f_\alpha$'s may be a proper subset of $\mathcal{F}$). The collection $\{f_\alpha\}$ is indexed by a subset $J'$ of the set $J$ and is thus finite. We will show that the open balls $B_\rho(f_\alpha, \varepsilon)$ for $\alpha \in J'$ cover $\mathcal{F}$.





## Lemma 45.3 (continued 2)

**Proof (continued).** Let $x \in X$. Choose $i \in \{1, 2, \ldots, k\}$ so that $x \in U_i$. Then

$$\begin{aligned} d(f(x), f(a_i)) &>< \delta \quad \text{since } x \in U_i \text{ implies } d(f(x), f(a_i)) < \delta \\ &\qquad \text{by the equicontinuity of } \mathcal{F} \\ d(f(a_i), f_\alpha(a_i)) &< \delta \quad \text{since } f(a_i), f_\alpha(a_i) \in V_{\alpha(i)} \text{ and } \operatorname{diam}(V_{\alpha(i)}) < \delta \\ d(f_\alpha(a_i), f_\alpha(x)) &>< \delta \quad \text{since } x \in U_i \text{ implies } d(f(x), f(a_i)) < \delta \\ &\qquad \text{by the equicontinuity of } \mathcal{F}. \end{aligned}$$

Hence

$$d(f(x), f_\alpha(x)) \le d(f(x), f(a_i)) + d(f(a_i), f_\alpha(a_i)) + d(f_\alpha(a_i), f_\alpha(x)) < 3\delta = \varepsilon.$$

Since $x \in X$ is arbitrary then $\rho(f, f_\alpha) = \max\{d(f(x), f_\alpha(x))\} < \varepsilon$. So $f \in B_\rho(f_\alpha, \varepsilon)$, as claimed. Since $f \in \mathcal{F}$ is arbitrary, then $\{B_\rho(f_\alpha, \varepsilon) \mid \alpha \in J'\}$ is a finite open covering of $\mathcal{F}$ with $\varepsilon$-balls. Since $\varepsilon > 0$ is arbitrary, then $\mathcal{F}$ is totally bounded under $\rho$ (and hence under $\overline{\rho}$ as well, as described above). □

## Theorem 45.4

**Theorem 45.4. The Classical Version of Ascoli's Theorem.** Let $X$ be a compact space. Let $(\mathbb{R}^n, d)$ denote Euclidean space in either the square metric or the Euclidean metric. Give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under $d$.

**Proof.** Since $X$ is compact, the sup metric $\rho(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}$ is defined, as observed in the proof of Lemma 45.3. Also observed in the proof of Lemma 45.3, for $\varepsilon < 1$, an $\varepsilon$-ball under the sup metric and uniform metric are equivalent. So the topology given by the sup metric is the same as the topology given by the uniform metric (namely, the uniform topology) on $\mathcal{C}(X, \mathbb{R}^n)$. Let $\mathcal{G}$ denote the closure of $\mathcal{F}$ in $\mathcal{C}(X, \mathbb{R}^n)$.





## Theorem 45.4 (continued 1)

**Proof (continued).** STEP 1. Suppose $\mathcal{F}$ has compact closure; that is, suppose $\mathcal{G}$ is compact. Then by Theorem 45.1, $\mathcal{G}$ is totally bounded under metrics $\rho$ and $\overline{\rho}$. Total boundedness of $\mathcal{G}$ implies equicontinuous under $d$ by Lemma 45.2. Compactness if $\mathcal{G}$ implies boundedness of $\mathcal{G}$ under $\rho$ (cover $\mathcal{G}$ with open balls of radius 1, extract a finite subcover to get a found on the diameter of $\mathcal{G}$). That is, for some $g \in \mathcal{G}$ and some $M \in \mathbb{N}$, $\mathcal{G} \subset B_\rho(g, M)$ or $\rho(f,g) < M$ for all $f \in \mathcal{G}$. So $\sup\{d(f(x), g(x)) \mid x \in X\} < M$ for all $f \in \mathcal{G}$; that is, for all $a \in X$ and all $f \in \mathcal{G}$ we have $d(f(a), g(a)) < M$ and so $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\} \subset B_d(g(a), M)$ and we have that $\mathcal{G}$ is pointwise bounded. Since we have shown that $\mathcal{G}$ is equicontinuous under $d$ and $\mathcal{G}$ is pointwise bounded under $d$, then $\mathcal{F} \subset \overline{\mathcal{F}} = \mathcal{G}$ is equicontinuous and pointwise bounded under $d$. This proves the "only if" part of the theorem.





## Theorem 45.4 (continued 2)

**Proof (continued).** STEP 2. We now show that if $\mathcal{F}$ is equicontinuous and pointwise bounded under $d$, then so is the closure of $\mathcal{F}$, $\mathcal{G} = \overline{\mathcal{F}}$.
Let $\mathcal{F}$ be equicontinuous and pointwise bounded under $d$. Let $x_0 \in X$ an dlet $\varepsilon > 0$. By the equicontinuity of $\mathcal{F}$, there is a neighborhood $U$ of $x_0$ such that $d(f(x), f(x_0)) < \varepsilon/3$ for all $x \in U$ and $f \in \mathcal{F}$. Given $g \in \mathcal{G} = \overline{\mathcal{F}}$, choose $f \in \mathcal{F}$ so that $\rho(f,g) < \varepsilon/3$ (see Theorem 17.4); that is, $d(f(x), g(x)) < \varepsilon/3$ for all $x \in X$. So by the Triangle Inequality,

$$d(g(x), g(x_0)) \le d(g(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), g(x_0)) < 3(\varepsilon/3) = \varepsilon$$

for all $x \in X$. Since $g$ is an arbitrary element of $\mathcal{G}$, then $\mathcal{G}$ is equicontinuous at $x_0$; since $x_0 \in X$ is arbitrary, then $\mathcal{G}$ is equicontinuous on $X$.





## Theorem 45.4 (continued 3)

**Proof (continued).** Next, for given $a \in X$, by the pointwise boundedness of $\mathcal{F}$, there is $M \in \mathbb{N}$ such that $\mathrm{diam}(\mathcal{F}_a) = \mathrm{diam}(\{f(a) \mid f \in \mathcal{F}\}) \le M$. Then for given $g, g' \in \mathcal{G} = \overline{\mathcal{F}}$ there are $f, f' \in \mathcal{F}$ such that $\rho(f,g) < 1$ and $\rho(f', g') < 1$. Then

$$d(g(a), g'(a)) \le d(g(a), f(a)) + d(f(a), f'(a)) + d(f'(a), g'(a)) \le 1 + M + 1 = M+2.$$

Since $g, g'$ are arbitrary elements of $\mathcal{G}$, then $\mathrm{diam}(\mathcal{G}_a) = \mathrm{diam}(\{g(a) \mid g \in \mathcal{G}\}) \le M + 2$. That is, $\mathcal{G}$ is pointwise bounded under $d$.

STEP 3. We now show that if $\mathcal{G} = \overline{\mathcal{F}}$ is equicontinuous and pointwise bounded, then there is a compact subspace $Y$ of $\mathbb{R}^n$ that contains the union of the sets $g(X)$ for $g \in \mathcal{G}$.
Let $\mathcal{G} = \overline{\mathcal{F}}$ be equicontinuous and pointwise bounded. For each $a \in X$, by equicontinuity, there is a neighborhood $U_a$ of $a$ such that $d(g(x), g(a)) < 1$ for all $x \in U_a$ and for all $g \in \mathcal{G}$.

## Theorem 45.4 (continued 4)

**Proof (continued).** Cover $X$ with such open $U_a$'s and the compactness of $X$ implies that there are open $U_{a_1}, U_{a_2}, \ldots, U_{a_k}$ covering $X$. Since $\mathcal{G}_{a_i} = \{g(a_i) \mid g \in \mathcal{G}\}$ is bounded by the pointwise boundedness hypothesis, then $\cup_{i=1}^k \mathcal{G}_{a_i}$ is also bounded; say $\cup_{i=1}^k \mathcal{G}_{a_i} \subset B(0,N) \subset \mathbb{R}^n$. Since $d(g(a),g(a)) < 1$ for all $x \in U_{a_i}$ for all $a_i$ then $d(g(x),g(a)) < 1$ for all $x \in J$ and hence $g(X) \subset B(0,N+1)$. Let $Y = \overline{B}(0,N+1)$. Then $Y$ is the desired compact subspace of $\mathbb{R}^n$ (by the Heine-Borel Theorem, Theorem 27.3).

STEP 4. Now for the "if" part of the theorem, we use STEPS 2 and 3. Suppose $\mathcal{F}$ is equicontinuous and pointwise bounded under $d$. Since $\mathcal{G} = \overline{\mathcal{F}}$ is a closed subspace of $(\mathcal{C}(X,\mathbb{R}^n),\rho)$ (recall that $\mathcal{F}$ is hypothesized to be a subspace of $\mathcal{C}(X,\mathbb{R}^n)$), and since $(\mathcal{C}(X,\mathbb{R}^n),\rho)$ is complete (by Theorem 43.6) then $\mathcal{G}$ is complete (a closed subset includes all limit points and so all Cauchy sequences converge in the closed subset).





## Theorem 45.4 (continued 5)

**Theorem 45.4. The Classical Version of Ascoli's Theorem.** Let $X$ be a compact space. Let $(\mathbb{R}^n, d)$ denote Euclidean space in either the square metric or the Euclidean metric. Give $\mathcal{C}(X,\mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X,\mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under $d$.

**Proof (continued).** By STEP 2, $\mathcal{G}$ is equicontinuous and pointwise bounded under $d$. By STEP 3, there is a compact subspace $Y$ of $\mathbb{R}^n$ such that $\cup\{g(X) \mid g \in \mathcal{G}\} \subset Y$; so $\mathcal{G} \subset \mathcal{C}(X,Y)$ where $Y$ is compact. By Lemma 45.3, $\mathcal{G}$ is totally bounded under $\rho$. By Theorem 45.1, since $\mathcal{G}$ is complete and totally bounded, then $\mathcal{G} = \overline{\mathcal{F}}$ is compact, That is, $\mathcal{F}$ has compact closure as claimed. □





## Corollary 45.5

**Corollary 45.5.** Let $X$ be compact. Let $d$ denote either the square metric or the Euclidean metric on $\mathbb{R}^n$. Give $\mathcal{C}(X,\mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X,\mathbb{R}^n)$ is compact if and only if it is closed, bounded under the sup metric $\rho$, and equicontinuous under $d$.

**Proof.** If $\mathcal{F}$ is compact then it is closed and bounded (bounded as argued in the proof of STEP 1 of Theorem 45.4 and closed since compact implies limit point compact by Theorem 28.1). So by Theorem 45.4, $\mathcal{F}$ is equicontinuous.

Conversely, if $\mathcal{F}$ is closed then $\mathcal{F} = \mathcal{G} = \overline{\mathcal{F}}$; if $\mathcal{F}$ is bounded under $\rho$, then it is pointwise bounded under $d$; and if $\mathcal{F}$ is also equicontinuous then Theorem 45.4 implies that $\mathcal{F}$ is compact. □