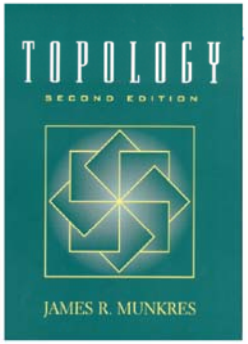
Introduction to Topology

Chapter 9. The Fundamental Group

Section 51. Homotopy of Paths—Proofs of Theorems



Lemma 51.1

Lemma 51.1. The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Proof. $\simeq$ and $\simeq_p$ are symmetric since $F(x,t) = f(x)$ is a homotopy (or path homotopy if f is a path).

Suppose $f \simeq f'$. Then there is a homotopy (or path homotopy) $F(x,t)$ between f and f'. Then $G(x,t) = F(x, 1-t)$ is a homotopy (or path homotopy) between f' and f; so $f' \simeq f$ (or $f' \simeq_p f$).

Suppose $f \simeq f'$ and $f' \simeq f''$. Then there is a homotopy F from f to f' and a homotopy F' between f' and f''. Define

$$G(x,t) = \begin{cases} F(x,2t) & \text{for } t \in [0,1/2] \\ F'(x,2t-1) & \text{for } (1/2,1]. \end{cases}$$

Then G is a homotopy between f and f''; so $f \simeq f''$. Similarly, $\simeq_p$ is transitive. □

Theorem 51.2

Theorem 51.2. The operation $*$ on the equivalence classes of paths in space X satisfies the following properties:

(1) Associativity: If $[f] * ([g] * [h])$ is defined, then so is $([f] * [g]) * [h]$ and they are equal.

(2) Right and left Identities: Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1 then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(3) Inverses: Given the path f in X from x_0 to x_1 let $\overline{f}$ be the path defined by $\overline{f}(s) = f(1-s)$. Then $\overline{f}$ is called the *reverse* of f, $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

Theorem 51.2 (continued)

Proof. Notice that for continuous $k : X \to Y$, if f and g are paths in X with $f(1) = g(0)$ then $k \circ (f * g) = (k \circ f) * (k \circ g)$ (*). That is, the image of $f * g$ under continuous mapping k is the image of f [illegible] the image of g.

For right and left identities, let e_0 denote the constant path in I at ($e_0(s) = 0$ for $s \in I$) and let $\iota : I \to I$ denote the identity map (which is a path in I from 0 to 1). Then $e_0 * I$ is also a path in I from 0 to 1.

Because I is convex there is a path homotopy G in I between ι and $e_0 * \iota$. Then for any f a path from x_0 to x_1 we have by (*) that

$$F \circ (e_0 * \iota) = (f \circ e_0) * (f \circ \iota) = e_{x_0} * f \qquad (1)$$

since $f \circ e_0(s) = f(0) = x_0$ for all $s \in [0,1]$, or $f \circ e_0 = e_{x_0}$ by the definition of e_{x_0} and $f \circ \iota = f$.

Theorem 51.2 (continued)

Now G is a path homotopy between ι and $e_0 * \iota$, so $f \circ G(s,t)$ gives us $f \circ G([illegible]) = f \circ \iota = f$ and $f \circ G(s,1) = f \circ (e_0 * \iota)$. So $f \circ G$ is a path homotopy between f and $f \circ (e_0 * \iota)$. That is, $f \cong \rho f \circ (e_0 * \iota)$.

So the product $e_{x_0} * f$ produces a path equivalent to f. So, by the Lemma 51.A, $[f] = [e_{x_0} * [f]$. Similarly, with e, the constant path at 1 and $e_{x_1} = f \circ e$, we get $[f] * [e_{x_1}] = [f]$.

For inverses, notice that the reverse ι in $\overline{\iota}(s) = 1 - s$. Then $i * \overline{\iota}$ is a path in I with initial and final point O. The constant path e_0 is also a path in I with initial and final point O.

Since I is convex, there is a path homotopy H in I between e_0 and $i * \overline{\iota}$.

Theorem 51.2 (continued)

Notice that $H(s,t)$ gives $f \circ G(s,0) = f \circ e_0 = e_{x_0}$ and $f \circ G(s,1) = f \circ (\iota * \overline{\iota}$, so $f \circ H$ is a path homotopy between e_{x_0} and $f \circ (\iota * \overline{\iota})$ where

$$f \circ (\iota * \overline{\iota} = (f \circ \iota) * (f \circ \iota) = f * \overline{f} \textit{ by } (*) \qquad (2)$$

By the definition of $\overline{f}$.

So $e_{x_0} \cong_p f * \overline{f}$. By Lemma 51.A, $[f] * [\overline{f}] = e_{x_0}$.

Similarly, by considering $\iota * \iota$, we can show that $[\overline{f}] * [f] = e_{x_1}$.

Theorem 51.2 (continued)

Now for associativity. It will be convenient to describe the product $f \times g$ in a different way. If $[a,b],[c,d]$ are two intervals in $\mathbb{R}$, there is a unique map $p : [a,b] \to [c,d]$ of the form $p(x) = mx + k$ where $p(a) = c$ and $p(b) = d$.

p is called the <u>positive linear map</u> of $[a,b]$ to $[c,d]$ (because its graph is a straight line with positive slope). The inverse of a positive linear map is a positive linear map and the composition of two such maps is such a map.

Now the product $f * g$ (which has domain $[0,1]$) can be described as follows: On $[0,\frac{1}{2}]$, it equals the positive linear map $[0,\frac{1}{2}]$ to $[0,1]$ followed by f; and on $[\frac{1}{2},1]$ it equals the positive linear map of $[\frac{1}{2},1]$ to $[0,1]$ followed by g.

Theorem 51.2 (continued)

Given paths f, g, and h in X, the products $f * (g * h)$ and $(f * g) * h$ are defined precisely when $f(1) = g(0)$ and $g(1) = h(0)$. With this as the case, we define the product as follows:

Choose a and b in I so that $0 < a < b < 1$. Define a path $k_{a,b}$ in X as follows: On $[0,a]$, $k_{a,b}$ equals the positive linear map of $[a,b]$ to I followed by g; and on $[b,1]$ it equals the positive linear map of $[b,1]$ to I followed by h.

We now show that if c and d are another pair of points of I with $0 < c < d < 1$, then $k_{c,d}$ is path homotopic to $k_{a,b}$. Let $p : I \to I$ be the continuous positive linear map mapping $[0,a] \to [0,c]$, $[a,b] \to [c,d]$, and $[b,1] \to [d,1]$.

Theorem 51.2 (continued)

Then $k_{c,d} \circ p = k_{a,b}$. But also, p is a path in I from 0 to 1. So there is a path homotopy P in I between p and ι.

Then $k_{c,d} \circ P$ is a path homotopy between $k_{a,b}$ and $k_{c,d}$. That is, $k_{a,b} \simeq_p k_{c,d}$.

Now by the definition of $*$, $f * (g * h) = k_{a,b}$ where $a = \frac{1}{2}$ and $b = \frac{3}{4}$ (so f is the first half and $g * h$ is the second half of the image of I) and $(f * h) * g = k_{c,d}$ where $c = \frac{1}{4}$ and $d = \frac{1}{2}$.

Hence $(f * g) * h \simeq_p f * (g * h)$ and by Lemma 51.A, $([f] * [g]) * [h] = [f] * ([g] * [h])$. □