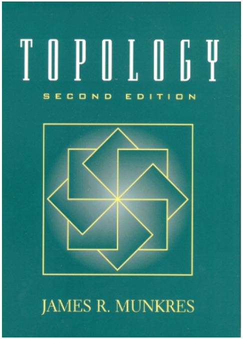
Introduction to Topology

Chapter 9. The Fundamental Group

Section 52. The Fundamental Group—Proofs of Theorems



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Theorem 52.1

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Proof. $\hat{\alpha}$ is a homomorphism since

$$\begin{aligned} \hat{\alpha}([f]) * \hat{\alpha}([g]) &= ([\overline{\alpha} * [f] * [\alpha]) * ([\overline{\alpha} * [g] * [\alpha]) \\ &= [\overline{\alpha}] * [f] * ([\alpha] * [\overline{\alpha}]) * [g] * [\alpha] \\ &= [\overline{\alpha}] * ([f] * [g]) * [\alpha] \\ &= \hat{\alpha}([f] * [g]) \end{aligned} \tag{1}$$

To show that $\hat{\alpha}$ is an isomoprhism, we show that it is one to one and onto by show that it has an inverse $\hat{\beta} : \pi_1(X, x_1) \to \pi_1(X, x_0)$. Define

$$\hat{\beta} = [\alpha] * [h] * [\overline{\alpha}] \tag{2}$$

(So $\hat{\beta}$ is based on $\beta = \overline{\alpha}$).

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$$\hat{\beta} = [\alpha] * [h] * [\overline{\alpha}] \tag{2}$$

(So $\hat{\beta}$ is based on $\beta = \overline{\alpha}$).

Theorem 52.1 continued

For each $[h] \in \pi_1(X, x_1)$ we have

$$\begin{aligned} \hat{\alpha}(\hat{\beta}([h])) &= \hat{\alpha}([\alpha] * [h] * [\overline{\alpha}]) \\ &= [\overline{\alpha}] * ([\alpha] * [h] * [\overline{\alpha}]) * [\alpha] = [h] \end{aligned} \tag{3}$$

For each $[f] \in \pi_1(X, x_0)$ we have

$$\begin{aligned} \hat{\beta}(\hat{\alpha}([f])) &= \hat{\beta}([\overline{\alpha}] * [f] * [\alpha]) \\ &= [\alpha] * ([\overline{\alpha}] * [f] * [\alpha]) * [\overline{\alpha}] = [f] \end{aligned} \tag{4}$$

Therefore $\hat{\alpha}$ is an isomorphism. $\square$

Theorem 52.1 continued

For each $[h] \in \pi_1(X, x_1)$ we have

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Therefore $\hat{\alpha}$ is an isomorphism. □

Lemma 52.3

Lemma 52.3. In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Proof. Let α and β be two paths from x_0 to x_1. Then $\alpha * \overline{\beta}$ is defined and is a loop on X based at x_0. Since X is simply connected, this loop is path homotopic to the constant loop at x_0 (by the definition of simply connected); i.e. $\alpha * \overline{\beta} \simeq_p e_{x_0}$.

Then

$$\begin{aligned}
[\alpha] &= [\alpha] * ([\overline{\beta}] * [\beta]) \\
&= ([\alpha] * [\overline{\beta}]) * [\beta] \quad \textit{by associativity} \\
&= [\alpha * \overline{\beta}] * [\beta] \quad \textit{by defn of } * \\
&= [e_{x_0}] * [\beta] \quad \textit{by above} \\
&= [\beta] \quad \textit{since } [e_{x_0}] \textit{ is the identity in } \pi_1(X, x_0)
\end{aligned} \tag{5}$$

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Theorem 52.4

Theorem 52.4. If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $\iota : (X, x_0) \to (X, x_0)$ is the identity map, then ι_* is the identity homomorphism.
Proof. By the definition of the induced homomorphism, $(k \circ h)_*([f]) = [(k \circ h) \circ f]$, and

$$\begin{aligned} (k_* \circ h_*)([f]) &= k_*(h_*([f])) \\ &= k_*([h \circ f]) \quad \textit{by defn of } h_* \\ &= [k \circ (h \circ f)] \quad \textit{by defn of } k_* \\ &= [(k \circ h) \circ f] \quad \textit{since function composition is associative}. \end{aligned} \tag{6}$$

So $(k \circ h)_* = k_* \circ h_*$.

Theorem 52.4 Continued

Similarly,

$$\begin{aligned} \iota_*([f]) &= [\iota \circ f] \quad \textit{by defn of } \iota_* \\ &= [f] \end{aligned} \tag{7}$$

and ι_* is the identity homomorphism. □

Corollary 52.5

Corollary 52.5. If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X and Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Proof. Since h is a homeomorphism, it has a continuous inverse (by definition), say it is $k : (Y, y_0) \to (X, x_0)$. Then by Theorem 52.4, $(k_* \circ h_*) = (k \circ h)_* = \iota_*$ where ι is the identity map of (X, x_0). Similarly, $(h_* \circ k_*) = (h \circ k)_* = j_*$ where j is the identity map of (Y, y_0).

Since ι_* and j_* are the identity homomorphisms (in fact, identity isomorphisms) of groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, and since $k_* \circ h_* = \iota_*$ and $h_* \circ k_* = j_*$, then k_* is the inverse of h_* and so h_* is a one to one and onto homomorphism. That is, h_* is a group isomorphism. $\square$

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Since ι_* and j_* are the identity homomorphisms (in fact, identity isomorphisms) of groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, and since $k_* \circ h_* = \iota_*$ and $h_* \circ k_* = j_*$, then k_* is the inverse of h_* and so h_* is a one to one and onto homomorphism. That is, h_* is a group isomorphism. □