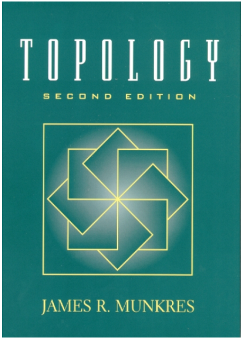
Introduction to Topology

Chapter 9. The Fundamental Group

Section 53. Covering Spaces—Proofs of Theorems



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Proof. Suppose $A \subseteq E$ is open and let $x \in p(A)$. Let U be a neighborhood of x that is evenly covered by p. Let $\{V_\alpha\}$ be the slices that partition $p^{-1}(U)$.

There is a point $y \in A$ such that $p(y) = x$; let V_β be the slice containing y (there is such a y in each V_β, but maybe only one in A).

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Theorem 53.1

Theorem 53.1. The map $p : \mathbb{R} \to S^1$ (the "1-sphere") given by the equation $p(x) = (cos(2\pi x), sin(2\pi x))$ is a covering map.

Proof. We need to find an open neighborhood of each point of S^1 that is evenly covered by p. We will use the four pieces of S^1 determined by intersecting it with the open upper, lower, left and right half plane. THe four open sets cover S^1 and, as we will show, are evenly covered.

Consider U_1, the open subset of S^1 consisting of all points lying in the right half plane. These have positive x coordinates when we treat S^1 as a subset of R^2.

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So U_1 is the image of $(n - \frac{1}{4}, n + \frac{1}{4}) \subseteq \mathbb{R}$ for any $n \in \mathbb{Z}$. So $p^{-1}(U - 1) = \cup_{n \in \mathbb{Z}}(n - \frac{1}{4}, n + \frac{1}{4})$.

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So the slices are $(V_n = (n - \frac{1}{4}, n + \frac{1}{4})$ where $n \in \mathbb{Z}$. Now p maps V_n ina a one to one, onto, continuous map (and has a continuous inverse). So p maps V_n homeomorphically to U_1.

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Theorem 53.2

Theorem 53.2. Let $p : E \to B$ be a covering map. If B_0 is a subspace of B_ι and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p to E_0 is a covering map of B_0.

Proof. Given $b_0 \in B_0$, Let U be an open set in B containing b_0 that is evenly covered by p. Let $\{V_2\}$ be a partition of $p^{-1}(u)$ into slices.

Then $U \cap B_0$ is a neighborhood of b_0 in B_0 and the sets $V_2 \cap E_0$ are disjoint open sets in E_0 where union is $p^{-1}(U \cap E_0$ and each $V_2 \cap E_0$ is mapped homeomorphically onto $U \cap B_0$ onto $U \cap B_0$ by p (a restriction of a homeomorphism is a homeomorphism).

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Proof. Given $b \in B$ and $b' \in B'$, let U and U' be neighborhoods of b and b', respectively, that are evenly covered by p and p', respectively. Let $\{V_2'\}$ and $\{V_B'\}$ be partitions of $p^{-1}(U)$ and $p'^{-1}(U')$, resp., into slices.

Then the inverse image under $p \times p'$ of the open set $U \times U'$ is the union of all the set $V_2 \times V_\beta'$.These are disjoint open sets of $E \times E'$ (since the V_2 are disjoint/open in E and the V_β are disjoint/open in E').

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